Wilsonian Effective Theory

K. G. Wilson PRB 4 (1971) 3174, 3184
K G Wilson & J. G. Kogut Phys Reports 12 (1974) 75
J. Polchinski, NPB 231 (1984) 269 and references therein

In the theory of critical phenomena, as a system approaches a 2nd order phase transition as a result of smoothly changing external macroscopic parameters (like temperature and applied magnetic field for a ferromagnet close to the Curie point), the long distance fluctuations dominate the behavior — the correlation length becomes very large, much larger than the scale of the microscopic physics that underlies the behavior. These long distance fluctuations are, in the quantum field theoretic language that we have been using, low energy (or low mass) modes. For critical phenomena the field fluctuations are thermal in character - unless at or close to $T=0$, quantum effect are subdominant. But the description of the system is accurately given by statistical mechanics through the computation of a partition function

$$Z = \int [d\phi]\, e^{-S[\phi,J]}$$

with J an external source. This is identical to the imaginary time version of the quantum field theory — and the reason we have used "S" for what should be βH ($\beta = 1/kT \to 1/\hbar$ in QFT, $H(\phi) \to S(\phi)$)
So we see again in this context that the very long distance physics depends weakly on the microscopic short distance physics. Is there a connection to effective lagrangians?

In CM, block spin transformations are used to uncover the dynamics of the very long distance modes. Block spin transformations take a model (say, a spin model like Ising, or XY, or Heisenberg) and define a new, equivalent partition function defined on a coarser lattice. The renormalization group is used to analyze the (infinite) sequence of applications of this transformation. If the RG evolution approaches a fixed point then the physics of the long distance modes is completely determined by the properties at the fixed point — reference to the underlying microscopic lattice dynamics is lost. Even away from the fixed point one may be able to infer approximately the behavior of long distance modes independently of short distance details (but only approximately, the is some small residual dependence on short distance properties of the model/physics).

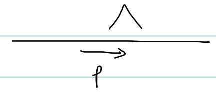
There is a continuum analog of this. We'd like to take a look at it because

(i) It illustrates these ideas in the continuum setting, showing they are more general

(ii) It sheds light into renormalization

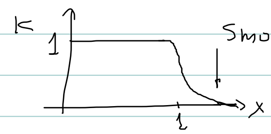
(iii) It gives another perspective to EFT.

Consider a theory in 4D (but we can easily do this in any D) that we render finite by imposing a momentum cutoff Λ. To be slightly more concrete we take the Euclidean action (ie, imaginary time) to be that of a scalar field with propagator

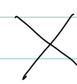


$$= \frac{K(p^2/\Lambda^2)}{p^2+m^2} \qquad p^2 = p_\mu p_\nu g^{\mu\nu} \qquad g^{\mu\nu} = (++++).$$

where $K(x)$



Suppose the theory contains a $\frac{1}{4!}\lambda\phi^4$ interaction term

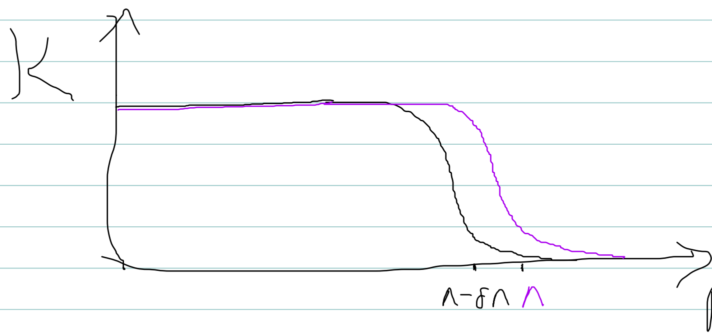
In Feynman diagrams:  $= \lambda$

We'd like to start from, say, this Lagrangean with some initial cutoff Λ_0 so call it $L(\Lambda_0)$, and ask how do we change, $L(\Lambda_0) \to L(\Lambda)$, in order to have the same Green functions if we reduce the cut-off to a scale $\Lambda < \Lambda_0$. For starters we note that this question does not even make sense for Green functions with at least one external leg with momentum p in the range $\Lambda < |p| < \Lambda_0$: the cutoff $K(p^2/\Lambda^2) = 0$ in this range so there is no hope of $L(\Lambda)$ reproducing $L(\Lambda_0)$.

So we will assume $|p| < \Lambda < \Lambda_0$ for all external legs.

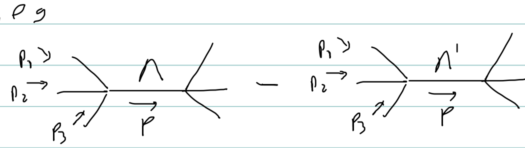
It will be easier to do this by assuming we know already $L(\Lambda)$ for some arbitrary Λ and we compute $L(\Lambda')$ with $\Lambda' = \Lambda - \delta\Lambda$. We can then go from Λ_0 to Λ by a sequence of infinitesimal changes.

So we are comparing Lagrangians with a cut-off difference:

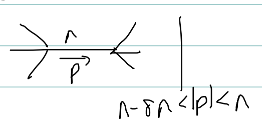


Now, we do not know the form of $L(\Lambda)$ (meaning the functional dependence on ϕ & $\partial_\mu\phi$). But we can be pretty sure it contains a $\frac{1}{4!}\lambda\phi^4$ term.

So consider, eg



Now, for $|p|<\Lambda'$ and $|p|>\Lambda$ they agree. Only in the narrow window $\Lambda'<|p|<\Lambda$ do we see a difference: the graph with $K(p/\Lambda')$ vanishes, while the other one is



$$\approx \lambda \cdot \frac{1}{\Lambda^2+m^2}\lambda \approx \frac{1}{\Lambda^2}\lambda^2$$

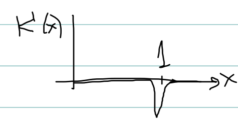
We can reproduce the effect of this in the effective theory if we introduce an interaction $\sim \phi^6$:

$$L(\Lambda-\delta\Lambda) - L(\Lambda) \sim \lambda^2 \frac{1}{\Lambda^2}\phi^6$$

More precisely (using $1/(\Lambda-\delta\Lambda)^2 = \frac{1}{\Lambda^2}(1+2\frac{\delta\Lambda}{\Lambda})$)

$$\text{diff} = \lambda^2 \frac{K(p^2/\Lambda^2)-K(p^2/(\Lambda-\delta\Lambda)^2)}{p^2+m^2} = \lambda^2 \cdot \frac{-2\frac{p^2}{\Lambda^2}\frac{\delta\Lambda}{\Lambda}\cdot K'(p^2/\Lambda^2)}{p^2+m^2}$$

and



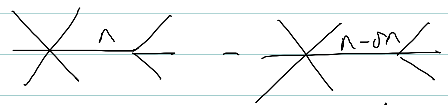
so

or $$L(\Lambda)-L(\Lambda-\delta\Lambda) \approx -2\frac{\lambda^2}{\Lambda^2}\frac{\delta\Lambda}{\Lambda}K'$$

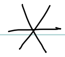
or $$\Lambda\frac{dL}{d\Lambda} \approx -2\lambda^2\frac{\phi^6}{\Lambda^2}$$ with $K'\approx 1$ since $p^2\approx\Lambda^2$

Integrating out high momentum modes is much like integrating out heavy massive particles?

Now, since $L(\Lambda)$ now has an interaction $\frac{g}{\Lambda^2}\phi^6$ in going to $L(\Lambda-\delta\Lambda)$ I should also consider this interaction. At tree level,

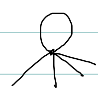


$\longrightarrow \quad \frac{1}{\Lambda^4}\phi^8$ interaction in L

Note that $\frac{1}{\Lambda^4}$ arises from $\frac{1}{\Lambda^2}$ in  and $\frac{1}{\Lambda^2}$ from difference at the edge of the momentum region, the shell $\Lambda-\delta\Lambda<|p|<\Lambda$, from the propagator as before

Clearly this goes on, generating $\frac{1}{\Lambda^{2n}}\phi^{4+2n}$ terms in $L(\Lambda)$. Only even powers because we started from a model with $\phi\to-\phi$ symmetry.

1-loop: Now



$$= \frac{g}{\Lambda^2}\int\frac{d^4p}{(2\pi)^4}\frac{K(p^2/\Lambda^2)}{p^2+m^2} \sim \left(\frac{g}{\Lambda^2}\right)\left(\frac{1}{16\pi^2}\Lambda^2\right)$$

So the effect of changing the cutoff is $\frac{g}{\Lambda^2}\frac{1}{16\pi^2}\left(\Lambda^2-(\Lambda-\delta\Lambda)^2\right)=\frac{g}{16\pi^2}\frac{2\delta\Lambda}{\Lambda}$

or $\Lambda\frac{dL}{d\Lambda}=2\frac{g}{16\pi^2}\phi^4$

Although I did not show this, derivative operators, like $\frac{1}{\Lambda^2}\phi^2\partial_\mu\phi\,\partial^\mu\phi$, etc. will be needed.

So what we have is that even if we start from $L(\Lambda_0) = Z_0 \frac{1}{2} \partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m_0^2\phi^2 + \frac{1}{4!}\lambda_0\phi^4$

to reproduce this microscopic law with a lower cut-off $\Lambda < \Lambda_0$, we need a more general lagrangean

$$L(\Lambda) = Z(\Lambda)\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m^2\phi^2 + \frac{1}{4!}g_4\phi^4 + \frac{g_6}{\Lambda^2}\phi^6 + \frac{g_8}{\Lambda^4}\phi^8 + \dots$$

and the couplings are determined by a set of equations for $\Lambda\frac{dg_4}{d\Lambda}$, $\Lambda\frac{dg_6}{d\Lambda}, \dots$

We will return to the derivation of $\Lambda\frac{dL}{d\Lambda}$ to make it slightly more precise, but let's first study the meaning of these flow equations, or rather of their solution, and its implication.

It will suffice to consider g_4 & g_6: the physics will be clear without the complication of an infinite set of equations.

The RGE equations give $\Lambda\frac{dg_n}{d\Lambda}$. They are obtained from $\Lambda\frac{dL}{d\Lambda}$ so more properly we should look at

$$\Lambda\frac{d}{d\Lambda}\left(g_4\right) \quad \text{and} \quad \Lambda\frac{d}{d\Lambda}\left(\frac{g_6}{\Lambda^2}\right) \qquad \left(\text{and} \dots \Lambda\frac{d}{d\Lambda}\left(\frac{g_{4+2n}}{\Lambda^{2n}}\right)\right)$$

set these equal to functions of dimensionless couplings, $\beta_{2n}(g_4, g_6, \dots)$ times powers of Λ to make up for dimensions:

$$\Lambda\frac{dg_4}{d\Lambda} = \beta_4(g_4, g_6)$$

$$\Lambda\frac{d}{d\Lambda}\left(\frac{g_6}{\Lambda^2}\right) = \frac{1}{\Lambda^2}\beta_6(g_4, g_6) \to \left(\Lambda\frac{d}{d\Lambda} - 2\right)g_6 = \beta_6(g_4, g_6)$$

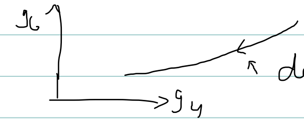
Before we analyze this system of 2 equations, note that for $\beta_6 = 0$ the solution is

$$g_6(\Lambda) = \left(\frac{\Lambda}{\Lambda_0}\right)^2 g_6(\Lambda_0) \implies \quad g_6 \text{ is quickly vanishing as } (\Lambda/\Lambda_0) \to 0$$

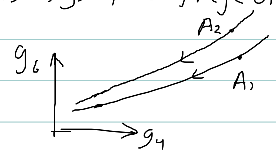
$$\left(\text{because } \frac{g_6(\Lambda)}{\Lambda^2} = \frac{g_6(\Lambda_0)}{\Lambda_0^2}\right)$$

Generally, all terms that $\to 0$ as $\Lambda \to 0$ (that is $g_n(\Lambda) \to 0$ as $(\frac{\Lambda}{\Lambda_0}) \to 0$) are called "irrelevant", while those that don't are called "relevant". Sometimes the distinction is made between more relevant operators that have $(\frac{\Lambda_0}{\Lambda})^n$ growth with $n>0$ and those for which the growth is logarithmic, $\ln \frac{\Lambda_0}{\Lambda}$, which are called "marginal", or "exactly marginal" if constant (Λ/Λ_0 independent).

Now we want to investigate flows in the g_4, g_6 plane



In fact we want to compare different trajectories, as they flow to long distances (IR, = small Λ). The point is that if the trajectories converge then the IR is insensitive to the UV. By converge I mean that, if we start at Λ_0 from two points A_1 & A_2 in the (g_4, g_6) plane, see Figure, then the flow to the IR brings the trajectories closer.



Let $\bar{g}_i$ be a flow (that is, a solution), and consider $\varepsilon_i = g_i - \bar{g}_i$. Then

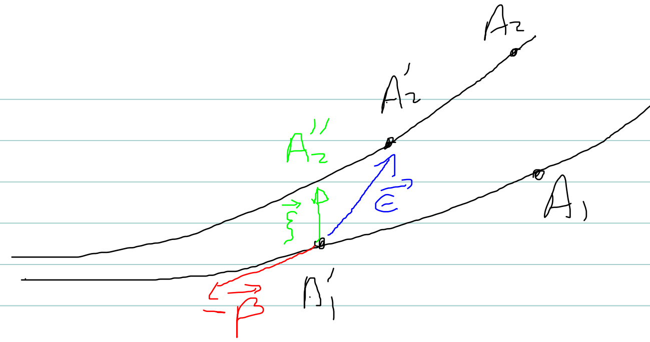
$$\Lambda \frac{d\varepsilon_4}{d\Lambda} = \beta_4(g_4, g_6) - \beta_4(\bar{g}_4, \bar{g}_6) = \varepsilon_4 \overline{\frac{\partial \beta_4}{\partial g_4}} + \varepsilon_6 \overline{\frac{\partial \beta_4}{\partial g_6}} \qquad \text{where } \overline{\frac{\partial \beta_i}{\partial g_k}} = \frac{\partial \beta_i(\bar{g}_4, \bar{g}_6)}{\partial \bar{g}_k}$$

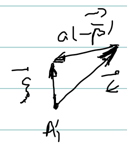
and

$$\Lambda \frac{d\varepsilon_6}{d\Lambda} - 2\varepsilon_6 = \varepsilon_4 \overline{\frac{\partial \beta_6}{\partial g_4}} + \varepsilon_6 \overline{\frac{\partial \beta_6}{\partial g_6}}$$

As A_1 & A_2 evolve, $\vec{\varepsilon} = (\varepsilon_4, \varepsilon_6)$ is a vector connecting the points.

We are not interested in the difference between the points A_1' & A_2', but rather, say, the difference between g_6 & $\bar{g}_6$ at a fixed g_4.





$(-\vec{\beta})$ gives the direction, need a constant a to make $\vec{\xi}$ vertical

$$\xi_4 = 0 = \epsilon_4 - a\bar{\beta}_4 \Rightarrow a = \epsilon_4/\beta_4$$

$$\text{so} \quad \xi_6 = \epsilon_6 - a\bar{\beta}_6 = \epsilon_6 - \epsilon_4 \bar{\beta}_6/\bar{\beta}_4$$

The evolution of ξ_6 is

$$\Lambda \frac{d\xi_6}{d\Lambda} - 2\xi_6 = \left[\frac{\partial\bar{\beta}_6}{\partial g_6} + \frac{\partial\bar{\beta}_4}{\partial g_4} - \Lambda\frac{d}{d\Lambda}\ln\bar{\beta}_4\right]\xi_6$$

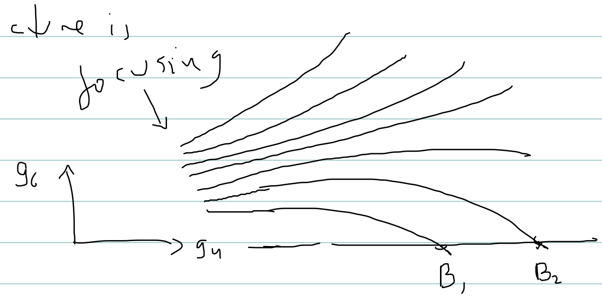
(Regarding this equation, see note on next page)

All this to show that not only the two trajectories have $g_6(\Lambda)$ decreasing (which we expect to happen from $g_6(\Lambda) \sim (\Lambda/\Lambda_0)^2$) but even more interesting, the distance between trajectories is squeezing them at a fast rate:

$$\xi_6(\Lambda) = \xi_6(\Lambda_0)\left(\frac{\Lambda^2}{\Lambda_0^2}\right)\left(\frac{\bar{\beta}_4(\Lambda_0)}{\bar{\beta}_4(\Lambda)}\right) \exp\left[\int_{\Lambda_0}^{\Lambda}\frac{d\Lambda'}{\Lambda'}\left(\frac{\partial\bar{\beta}_6}{\partial g_6} + \frac{\partial\bar{\beta}_4}{\partial g_4}\right)\right]$$

So to the extent that $g_{4,6}$ remain perturbative, or at least don't introduce growth with $\left(\frac{\Lambda_0}{\Lambda}\right)^x$ with a power $x \geq 2$, then $\xi_6(\Lambda)/\xi_6(\Lambda_0) \to 0$ rapidly as $\frac{\Lambda}{\Lambda_0} \to 0$.

So the picture is



As $\Lambda \to 0$, at any g_4, there is a single g_6, or rather a small range of values that gets smaller as Λ decreases $\Rightarrow$ the details of the microscopic theory are lost, approximately.

DIGRESSION

The equation $\Lambda \frac{d\xi_6}{d\Lambda} - 2\xi_6 = \left[\frac{\partial\bar\beta_6}{\partial g_6} + \frac{\partial\bar\beta_4}{\partial g_4} - \Lambda\frac{d}{d\Lambda}\ln\bar\beta_4\right]\xi_6$ was copied from Polchinski. I cannot reproduce it. Takin $d/d\ln\Lambda$ of ξ_6 I find

$$\Lambda\frac{d\xi_6}{d\Lambda} = \Lambda\frac{d}{d\Lambda}\left(\varepsilon_6 - \varepsilon_4\frac{\bar\beta_6}{\bar\beta_4}\right)$$

$$= 2\varepsilon_6 + \varepsilon_4\partial_4\bar\beta_6 + \varepsilon_6\partial_6\bar\beta_6 - \left(\varepsilon_4\partial_4\bar\beta_4 + \varepsilon_6\partial_6\bar\beta_4\right)\frac{\bar\beta_6}{\bar\beta_4}$$
$$- \frac{\varepsilon_4}{\bar\beta_4}\left(\bar\beta_4\partial_4\bar\beta_6 + \bar\beta_6\partial_6\bar\beta_6\right) + \frac{\varepsilon_4\bar\beta_6}{\bar\beta_4^2}\left(\bar\beta_4\partial_4\bar\beta_4 + \bar\beta_6\partial_6\bar\beta_4\right)$$

$$= 2\xi_6 + \varepsilon_4\left(2\frac{\bar\beta_6}{\bar\beta_4} + \cancel{\partial_4\bar\beta_6} - \cancel{\frac{\bar\beta_6}{\bar\beta_4}\partial_4\bar\beta_4} - \cancel{\partial_4\bar\beta_6} - \frac{\bar\beta_6}{\bar\beta_4}\partial_6\bar\beta_6 + \cancel{\frac{\bar\beta_6}{\bar\beta_4}\partial_4\bar\beta_4} + \frac{\bar\beta_6^2}{\bar\beta_4^2}\partial_6\bar\beta_4\right)$$
$$+ \varepsilon_6\left(\partial_6\bar\beta_6 - \frac{\bar\beta_6}{\bar\beta_4}\partial_6\bar\beta_4\right)$$

$$= 2\xi_6 + \varepsilon_4\frac{\bar\beta_6}{\bar\beta_4}\left(2 - \partial_6\bar\beta_6 + \frac{\bar\beta_6}{\bar\beta_4}\partial_6\bar\beta_4\right) + \varepsilon_6\left(\partial_6\bar\beta_6 - \frac{\bar\beta_6}{\bar\beta_4}\partial_6\bar\beta_4\right)$$

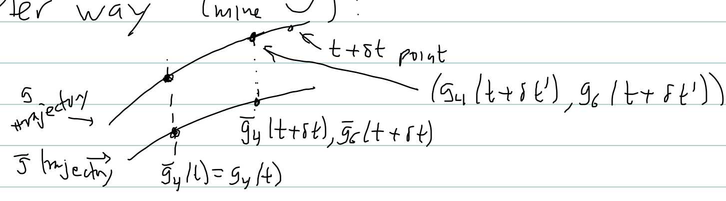
$$= 2\xi_6 + \left(\partial_6\bar\beta_6 - \frac{\bar\beta_6}{\bar\beta_4}\partial_6\bar\beta_4\right)\xi_6 + 2\varepsilon_4\frac{\bar\beta_6}{\bar\beta_4}$$

and use $\frac{d\ln\bar\beta_4}{d\ln\Lambda} = \partial_4\bar\beta_4 + \frac{\bar\beta_6\partial_6\bar\beta_4}{\bar\beta_4}$ $\rightarrow$

$$= 2\xi_6 + \left(\partial_6\bar\beta_6 + \partial_4\bar\beta_4 - \Lambda\frac{d}{d\Lambda}\ln\bar\beta_4\right)\xi_6 + 2\varepsilon_4\frac{\bar\beta_6}{\bar\beta_4}$$

which differs from Polchinski's by the last term.

Better way (mine ☺):



with $\delta t'$ = function of t and δt such that $g_4(t+\delta t') = \bar g_4(t+\delta t)$

$$g_4(t) + \delta t'\beta_4(g_4(t), g_6(t)) = \bar g_4(t) + \delta t\,\underbrace{\beta_4(\bar g_4(t), \bar g_6(t))}_{``\bar\beta_4"} \quad\Rightarrow\quad \delta t' = \frac{\bar\beta_4}{\beta_4}\delta t$$

equal

And $\xi_6 = g_6(t) - \bar g_6(t)$ evolves to $\xi_6(t+\delta t) = g_6(t+\delta t') - \bar g_6(t+\delta t)$

So $$\Lambda\frac{d\xi_6}{d\Lambda} = \frac{\bar\beta_4}{\beta_4}\left(2g_6 + \beta_6\right) - \left(2\bar g_6 + \bar\beta_6\right) = \frac{\bar\beta_4}{\bar\beta_4 + \xi_6\partial_6\bar\beta_4}\left(2(\bar g_6 + \xi_6) + \bar\beta_6 + \xi_6\partial_6\bar\beta_6\right) - 2\bar g_6 - \bar\beta_6$$

$$= \xi_6\left(2 + \partial_6\bar\beta_6\right) - \xi_6\partial_6\ln\bar\beta_4\left(2\bar g_6 + \bar\beta_6\right) + O(\xi_6^2)$$

$$= \left(2 + \partial_6\bar\beta_6 - 2\bar g_6\partial_6\ln\bar\beta_4 - \bar\beta_6\frac{\partial_6\bar\beta_4}{\bar\beta_4}\right)\xi_6$$

$$= \left(2 + \partial_4\bar\beta_4 + \partial_6\bar\beta_6 - \Lambda\frac{d}{d\Lambda}\ln\bar\beta_4 - 2\bar g_6\partial_6\ln\bar\beta_4\right)\xi_6$$

This is still different from Polchinski but at least it is homogeneous in ξ_6. Moreover, the extra term,

$$-2\bar{g}_6\, \partial_c \ln \bar{\beta}_4$$

has an explicit factor of $\bar{g}_6$ which is $\sim \mathcal{O}\left(\frac{\tilde{n}}{\tilde{n}_0}\right)$ so quickly becoming negligible compared with the 2.

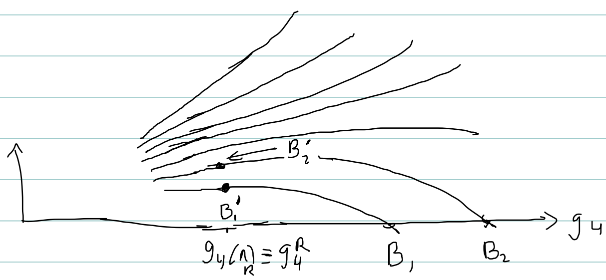
End of Digression

This does not mean that one can ignore $g_6(\Lambda)$ (and g_8 and so on) but rather that it is fixed → once we know $g_4(\Lambda)$.

Relation to renormalization and renormalizability.

In the picture B_1 is a point with $g_4(\Lambda_0)$ given and $g_6(\Lambda_0)=0$ (and $g_8(\Lambda_0)=0$ etc)

At $\Lambda_R << \Lambda_0$ the trajectory through B_1 gives a point B_1' defined to be g_4^R : $g_4(\Lambda_R)=g_4^R$



Now consider $\Lambda_0' > \Lambda_0$: there is another (longer) trajectory starting from B_2 that now gets to g_4^R at B_2'.

This procedure defines a bare coupling $g_4^B = g_4^B(g_4^R, \Lambda_R, \Lambda_0)$.

Consider now the limit of removing the cutoff keeping the renormalized quantities fixed, $\Lambda_0 \to \infty$, g_4^R = fixed, Λ^R = fixed.

$$\rightarrow \quad g_6(\Lambda_R) = g_6(\Lambda_0)\left(\frac{\Lambda_R^2}{\Lambda_0^2}\right) \times \cdots \longrightarrow 0 \quad \text{as } \Lambda_0 \to \infty$$

This is the usual statement of renormalization.

To get an equation for $\Lambda \frac{\partial S}{\partial \Lambda}$ we proceed as follows (following Polchinski's work)

Let

$$Z(J,\Lambda) = \int [d\phi]\, e^{\int \frac{d^4p}{(2\pi)^4}\left[-\frac{1}{2}\phi(p)\phi(-p)\Delta(p,\Lambda) + J(p)\phi(-p)\right] + \tilde{S}(\phi,\Lambda)} = \int [d\phi]\, e^{S(\phi,\Lambda)}$$

where $\Delta(p,\Lambda) = \frac{p^2+m^2}{K(p^2/\Lambda^2)}$ and $J(p)=0$ for $|p| \geq \Lambda$

$$\Rightarrow \Lambda \frac{dZ}{d\Lambda} = \int [d\phi]\, e^S \left[\int \frac{d^4p}{(2\pi)^4}\left(-\frac{1}{2}\phi(p)\phi(-p)(p^2+m^2)\,\Lambda \frac{\partial K^{-1}}{\partial \Lambda}\right) + \Lambda \frac{\partial \tilde{S}}{\partial \Lambda}\right] \qquad (\cancel{A})$$

To turn this into a useful equation for $\Lambda \frac{\partial \tilde{S}}{\partial \Lambda}$ we want to express the 1st term as a functional derivative operator on the action.

To get two powers of ϕ as the argument of the integrand consider

$$\frac{\delta}{\delta \phi(p_1)} e^S = e^S \frac{\delta S}{\delta \phi(p_1)}, \quad \frac{\delta^2 e^S}{\delta\phi(p_1)\delta\phi(p_2)} = e^S\left[\frac{\delta^2 S}{\delta\phi(p_1)\delta\phi(p_2)} + \frac{\delta S}{\delta\phi(p_1)}\frac{\delta S}{\delta\phi(p_2)}\right]$$

The $S-\tilde{S}$ part of this has a J term that we will be able to ignore, as it turns out, because $\int d^4p\, K(p^2/\Lambda^2) J(p)(\cdots) = 0$, since $J(|p|>\Lambda)=0$.

The rest involves

$$\frac{\delta(S-\tilde{S})}{\delta\phi(p)} = \frac{1}{(2\pi)^4}\phi(-p)\Delta(p) \Rightarrow \frac{\delta(S-\tilde{S})}{\delta\phi(p)}\frac{\delta(S-\tilde{S})}{\delta\phi(-p)} = \frac{1}{(2\pi)^4}\phi(-p)\phi(p)\Delta \cdot \frac{\Delta}{(2\pi)^4}$$

So that the 1st term in $(\cancel{A})$ is

$$\int d^4p \left(\frac{(2\pi)^4}{\Delta}\right) \Lambda \frac{\partial K^{-1}}{\partial \Lambda} \frac{\delta(S-\tilde{S})}{\delta\phi(p)} \frac{\delta(S-\tilde{S})}{\delta\phi(-p)}$$

A little algebra then shows that if

$$\Lambda \frac{\partial \tilde{S}}{\partial \Lambda} = -\frac{1}{2}\int d^4p\,(2\pi)^4 \frac{1}{p^2+m^2}\Lambda \frac{\partial K}{\partial \Lambda}\left[\frac{\delta \tilde{S}}{\delta\phi(p)}\frac{\delta \tilde{S}}{\delta\phi(-p)} + \frac{\delta^2 \tilde{S}}{\delta\phi(p)\delta\phi(-p)}\right] \qquad (\cancel{A}\cancel{A})$$

Then $\Lambda \frac{dZ}{d\Lambda} = 0$

Eq. $(\cancel{AA})$ is the more general version of the equation for $\Lambda \frac{dg_{4,c}}{d\Lambda}$ we wrote earlier.

Note: I have not verified fully the claim, but its almost there!

Bottom up EFT

Now that we know how to construct $\mathcal{L}_{eff}$ from $\mathcal{L}_{full}$, we can flip the question: can we construct $\mathcal{L}_{full}$ from $\mathcal{L}_{eff}$? The answer is probably no, there surely are many $\mathcal{L}_{full}$ for each $\mathcal{L}_{eff}$ (but if sufficient information exists about higher dim terms in $\mathcal{L}_{eff}$, I suspect different $\mathcal{L}_{full}$'s are equivalent. Still there is no efficient algorithm to do this. In practice

either $E \ll M$, $\rightarrow$ $(E/M)^n$ suppressed, $\frac{1}{M^n}\underset{\uparrow \dim \mathcal{O}}{\mathcal{O}^{(4+n)}}$ difficult to infer,

except for leading n that produces new effect, ie, lowest n for which $\mathcal{O}^{(4+n)}$ breaks an accidental symmetry of $\mathcal{L}_{eff}^{(\leq 4)}$.

Examples:

- B, L numbers in SM (from full = GUT)
- P, flavor in QCD + QED (from full = EW model = SM)

or $E \simeq M$ $\rightarrow$ sensitive to all powers of (E/M) $\rightarrow$ $\mathcal{L}_{eff}$ breaks down

$\rightarrow$ Look directly for effects of $\mathcal{L}_{full}$ (ie, spectrum/resonances)

The trick here is E just below threshold for new particle production. Even for $E \lesssim \frac{1}{2}M$ $\mathcal{L}_{eff}$ may help us organize our observations even if not completely systematically.

Complete later:

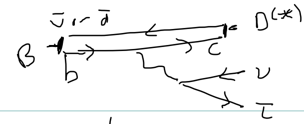
- $\mathcal{L}_{full}$ is unknown

$\longrightarrow$ $\mathcal{L}_{eff}$ = most gral consistent with symmetries

$\longrightarrow$ flows can be computed, but not matching (to what)?

$\rightarrow$ Still, if M of NP ("new physics") large enough, but not effectively ∞ can be useful in characterizing and constraining possible NP effects

Example: $B \to D^{(*)}\tau\nu$ anomalies



There are hints that $b \to c\tau\nu$ is different from $b \to c l \nu$, $l = e, \mu$

This breaks lepton flavor universality, an automatic property of the SM (that the weak & EM interactions of charged leptons have the same strength, modulo kinematic effects arising from their different masses).

In SM

$$\mathcal{L}_{\substack{EW\\ \text{theory}}} = -\frac{4G_F V_{cb}}{\sqrt{2}}\, \bar{\tau}_L \gamma^\mu \nu_L\, \bar{c}_L \gamma_\mu b_L + \text{h.c.}$$

Most general dim 6 set of ops with these quantum numbers
I do not care about whether they mix with each other under renormalization: I am trying to understand their possibly observable effects in $B \to D^{(*)}\tau\nu$. So (assuming neutrinos are left handed)

$$\mathcal{L}_{eff} = -\frac{4G_F V_{cb}}{\sqrt{2}} \Big[(1 + C_L)\, \bar{\tau}_L \gamma^\mu \nu_L\, \bar{c}_L \gamma_\mu b_L + C_R\, \bar{\tau}_L \gamma^\mu \nu_L\, \bar{c}_R \gamma_\mu b_R$$
$$+ C_{S_L}\, \bar{\tau}_R \nu_L\, \bar{c}_R b_L + C_{S_R}\, \bar{\tau}_R \nu_L\, \bar{c}_L b_R + C_T\, \bar{\tau}_R \sigma^{\mu\nu} \nu_L\, \bar{c}_R \sigma_{\mu\nu} b_L \Big] + \text{h.c.}$$

This is a fairly agnostic way to characterize the NP that arises from distance scales shorter than $\sim 1/m_b$.

But what if we assume the SM is embedded in the NP model and that there is a NP scale M below which $\mathcal{L}_{eff}$ is the SM plus dim ≥ 5 operators?

Then the dim 6 operators we are considering have to arise from dim 6 operators in the SM, ie. invariant under the EW group.

Recall SM: $SU(3) \times SU(2) \times U(1)$ reps: $(SU(3), SU(2))_{U(1)}$

q_L $(3,2)_{1/6}$
u_R $(3,1)_{2/3}$
d_R $(3,1)_{-1/3}$
l_L $(1,2)_{-1/2}$
e_R $(1,1)_{-1}$
H $(1,2)_{1/2}$
(and $\tilde{H} = \epsilon H^*$ $(1,2)_{-1/2}$)

Fierz: can choose always $(\overline{\text{quark}} \times \text{quark}) \cdot (\overline{\text{lepton}} \times \text{lepton})$

$\bar{q}_L \gamma^\mu q_L \, \bar{l}_L \gamma_\mu l_L \rightarrow$ neutral $\times$

$\bar{q}_L \gamma^\mu \tau^a q_L \, \bar{l}_L \gamma^\mu \tau^a l_L \rightarrow C_L$

$\underbrace{\bar{q}_L \, u_R}_{-1/6 \;\; 2/3 \;=\; 1/2}$ $\underbrace{\bar{l}_L e_R}_{1/2 \;\; -1 \;=\; -1/2}$ or $\underbrace{\bar{e}_R l_L}_{+1/2 \; \times}$ $\rightarrow \bar{q}_L u_R \bar{l}_L e_R$
or $\bar{u}_R q_L \bar{e}_R l_L \rightarrow C_{S_L}$

and $\bar{u}_R \sigma^{\mu\nu} q_L \bar{e}_R \sigma_{\mu\nu} l_L \rightarrow C_T$

$\bar{q}_L d_R \, \bar{e}_R l_L \rightarrow C_{S_R}$
$-\frac{1}{6} \; -\frac{1}{3} \quad 1 \; -1/2$

But no C_R: the leptonic left handed current $\bar{l}_L \gamma^\mu l_L$ is neutral and the "charged" current $\bar{l}_L \tau^a \gamma^\mu l_L$ is a 3 under $SU(2)$ but we cannot make a 3 out of right handed quarks.

Moreover, this predicts <u>correlations</u>: for example

$C_{S_R} \left(\bar{\tau}_R \nu_L \bar{c}_L b_R + \bar{\tau}_R \tau_L \bar{s}_L b_R \right)$ (or possibly $\tau_L \rightarrow e_L, \mu_L$ or a linear combination)

gives a NP contribution to $b \rightarrow s \tau^+ \tau^-$ (or $s \mu^+ \tau^-$ or $s e^+ \tau^-$) with the same strength, dictated by the coefficient C_{S_R}.

This suggests it may be of some use to list all the operators of this type of EFT $\Rightarrow$ the "SMEFT"

At dim 5 there is a unique ! term in $\mathcal{L}_{SMEFT}$

$$\frac{c_5}{M} \; H H \, \overline{l^c_R} l_L \;(+h.c.) \text{ "Weinberg operator"}$$

Here $l^c_R = C\, l_L^*$ where $* = (+)^T$ so $\overline{l^c_R}$ has the same quantum numbers as l_L.
The operator breaks "lepton number":
U(1) sym of SM with $l_L \to e^{i\alpha} l_L$ $e_R \to e^{i\alpha} e_R$, all else neutral.

Shifting the higgs $H \to \binom{0}{v/\sqrt{2}} + h$ produces

$$\frac{c_5 v^2}{M} \overline{\nu^c_R} \nu_L \qquad \text{a Majorana mass term for neutrinos.}$$

At dim 6 there are 2499 terms ! The number becomes more mangeable, ~50, if one counts without distinguishing generations, that is, for example

$$\bar{q}_{L i} \gamma^\mu q_{Lj} \, \bar{U}_{Rm} \gamma_\mu U_{Rn} \quad \text{counts as 1 rather than } 3^4.$$

The 1-loop 2499×2499 anomalous dimension matrix for these operators has been computed (by Rodrigo Alonso, a former student at UAM-IFT).

Unitarity + causality constraints

So far we have no a priori constrain on the Wilson Coefficients, ie, the coupling constants associated with each operator in $\mathcal{L}_{EFT}$:

$$\mathcal{L}_{EFT} = \mathcal{L}^{(4)} + \sum_n \underbrace{c_n}_{\text{arbitrary}} \mathcal{O}^{(n)}$$

We show through an example how some constraints may be placed on some c_n's using very general properties of QFT.

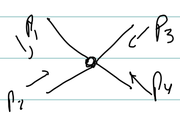
The example is an EFT of a single scalar field $\phi(x)$, invariant under constant shifts, $\phi(x) \to \phi(x) + a$

You probably recognize this as a Nambu-Goldstone boson.

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{c_1}{2M^4}(\partial_\mu \phi)^2(\partial_\nu \phi)^2 + \frac{c_2}{M^8}(\partial_\mu \phi)^2(\partial_\nu \phi)^2(\partial_\lambda \phi)^2 + \cdots$$

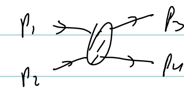
valid for $|p_\mu| \ll M$. Consider $2\to 2$ scattering

The Feynman rule is



$$= 4\frac{c_1}{M^4}\; p_1\cdot p_2\, p_3\cdot p_4 + p_1\cdot p_3\, p_2\cdot p_4 + p_1\cdot p_4\, p_2\cdot p_3$$

In terms of Mandelstam variables the amplitude



is

$$s = (p_1+p_2)^2 \quad t = (p_3-p_1)^2 \quad u = (p_4-p_1)^2$$

$$a(s,t,u) = \frac{c_1}{M^4}(s^2+t^2+u^2) + \cdots$$

(Of course $s+t+u=0$ so only s & t are independent):

$$a(s,t) = \frac{c_1}{M^4}\left(s^2+t^2+(s+t)^2\right)$$

Consider this as a function of s for fixed t.

Now, we want to gain information from the full theory, just using the basic fact that it is a consistent QFT (causal, local, unitary) $\Rightarrow$ optical theorem $\Rightarrow \mathrm{Im}\,a(s,0) > 0$ ($\mathrm{Im}\,a(s,0) = \#\, s\,\sigma_{Tot}$).

And to relate $\text{Im}\, a(s,0)$ to $a(s,t)$ at small s we may use a dispersion relation.

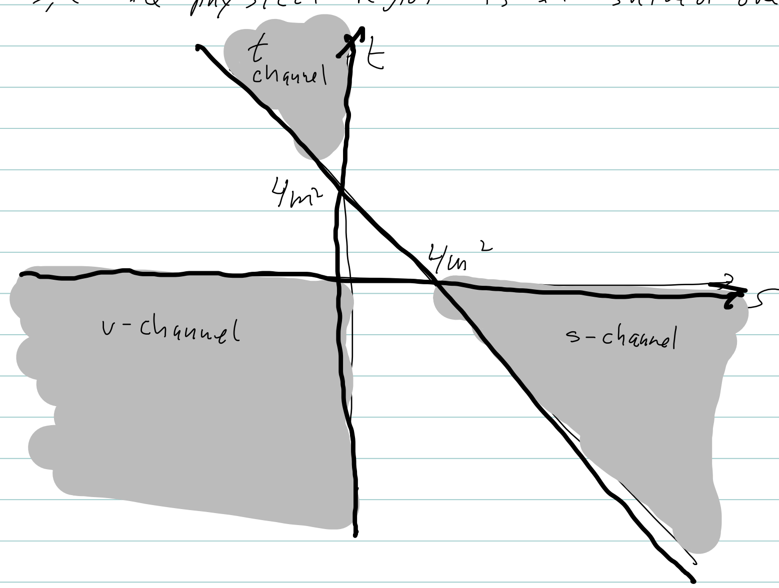
For this we need to understand where $a(s,0)$, when consider as a function of complex argument $a(z,0)$, $z \in \mathbb{C}$, has cuts, poles, etc.

ie, need analytic structure of $a(z,0)$.

We can't review this here, but in essence, cuts in $a(z,0)$ are at $z = s \in \mathbb{R}$ such that s corresponds to a physical process.

To study this we will add for now a mass term to $\mathcal{L}$, $-\frac{1}{2}m^2\phi^2$, so that ϕ has mass m.

For general s,t the physical region is the shaded one here:



To see this, consider s-channel, $1,2 \to 3,4$, in CM:

$$p_1 = (\sqrt{m^2+p^2}, \vec{p}),\ p_2 = (\sqrt{m^2+p^2}, -\vec{p}),\ p_3 = (\sqrt{m^2+p^2}, p\hat{n})\quad p_4 = (\sqrt{m^2+p^2}, -p\hat{n})$$

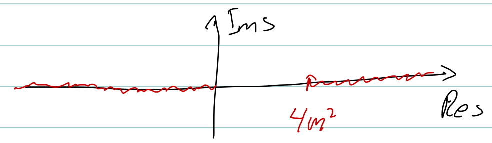
with $\vec{p}\cdot\hat{n}/p \equiv \cos\theta$, the scattering angle.

Then $s = 4(m^2+p^2) \geq 4m^2$ and $t = -2p^2(1-\cos\theta) < 0$.

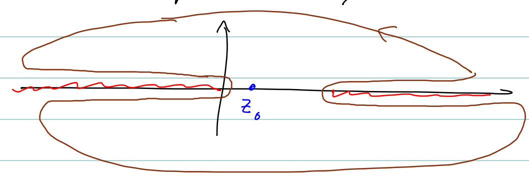
$u = -2p^2(1+\cos\theta) < 0 \Rightarrow 4m^2 - s - t < 0$

NOTE: In this simple case, no additional masses $\rightarrow$ $\left\{ \begin{array}{l} \text{branch points} \\ \text{are at} \\ 0, 4m^2 \end{array}\right.$

For $t=0$, in the complex s plane



Basic complex analysis



$$\frac{1}{2\pi i}\oint_C dz\,\frac{f(z)}{z-z_0} = f(z_0)$$

for $f(z)$ analytic per figure.

Now as semicircles to ∞, $\lim_{R\to\infty}\int \frac{Re^{i\phi} i\,d\phi}{Re^{i\phi}-z_0} f(Re^{i\phi}) \sim i\int d\phi\, f(Re^{i\phi})$

So if $f(|z|)\to 0$ as $|z|\to\infty$ we ignore this contribution:

$$f(z_0) = \frac{1}{2\pi i}\left[\int_{-\infty}^{0}\frac{dx}{x-z_0}\left[f(x+i\epsilon)-f(x-i\epsilon)\right] + \int_{4m^2}^{\infty}\frac{dx}{x-z_0}\left[f(x+i\epsilon)-f(x-i\epsilon)\right]\right]$$

Greens reflection: $f(x-i\epsilon) = f^*(x+i\epsilon)$

$$f(z_0) = \frac{1}{\pi}\int_{-\infty}^{0} + \int_{4m^2}^{\infty} dx\,\frac{\mathrm{Im}\, f(x+i\epsilon)}{x-z_0}$$

Let's use this to give us $a(s,0)$ for small s in terms of $\mathrm{Im}\, a(s,0) > 0$!

Things to take care of:

(1) $a(s,0)$ as $s\to\infty$?

(2) What is $\mathrm{Im}\, a(s,0)$ for $s<0$

(3) Sign of $\frac{1}{x-z_0}$

(4) $m\to 0$ (and we'll also do $z_0\to 0$).

Once we do we'll get ($m\to 0$ later)

$$\frac{d^2 a(s,0)}{ds^2} = \frac{2}{\pi}\int_{4m^2}^{\infty} dx\,\sqrt{x(x-4m^2)}\left[\frac{\sigma_s(x)}{(x-s)^3} + \frac{\sigma_u(x)}{(x-4m^2+s)^3}\right]$$

$\sigma_{s,u}$ = cross section in s & u channels. Hence $\frac{d^2 a(s,0)}{ds^2} > 0$

One may evaluate at $s=0$ to avoid s^n corrections $\Rightarrow \frac{c_1}{M^4} > 0 \Rightarrow \boxed{c_1 > 0}$

Now fill in details

(1) Froissart bound $|a(s,0)| < \# \, s \ln^2 s$

and $\frac{d}{ds} s\ln^2 s = \ln^2 s + 2\ln s$

$\frac{d^2}{ds^2} s\ln s = \frac{2\ln s}{s} + \frac{2}{s} \to 0$ as $s \to \infty$

$\Rightarrow$ Need $\frac{d^2 a}{ds^2}$

$$\Rightarrow \quad f'(z_0) = \frac{1}{\pi} \int\!\!+\!\!\int dx \frac{\mathrm{Im} f(x)}{(x-z_0)^2} \quad , \quad f''(z_0) = \frac{2}{\pi} \int_{-\infty}^{0} + \int_{4m^2}^{\infty} dx \frac{\mathrm{Im} f(x)}{(x-z_0)^3}$$

(3) and the rest of (2)

For first integral, $y = 4m^2 - x \Rightarrow$

$$f''(z_0) = \frac{2}{\pi} \int_{4m^2}^{\infty} dx \left[\frac{\mathrm{Im} f(x)}{(x-z_0)^3} - \frac{\mathrm{Im} f(4m^2-x)}{(x+z_0-4m^2)^3} \right]$$

or $$\frac{d^2 a(s,0)}{ds^2} = \frac{2}{\pi} \int_{4m^2}^{\infty} dx \left[\frac{\mathrm{Im}\, a(s+i\epsilon,0)}{(x-s)^3} - \frac{\mathrm{Im}\, a(4m^2-s+i\epsilon,0)}{(x+s-4m^2)^3} \right]$$

For optical theorem need $s+i\epsilon$ rather than $s-i\epsilon$

Optical theorem

$$\mathrm{Im}\, a(s+i\epsilon,0) = \sqrt{s(4m^2-s)}\, \sigma_s(s)$$

$$\mathrm{Im}\, a(4m^2-s+i\epsilon) = -\mathrm{Im}\, a(4m^2-s-i\epsilon) = -\mathrm{Im}\, a(u+i\epsilon, 0) = \sqrt{s(4m^2-s)}\, \sigma_u(s)$$

(In our case, in fact, crossing symmetry gives $\sigma_u = \sigma_s$).

DONE?

Note: we really should write $a = \frac{4c_1}{M^4}(p_1\cdot p_2\, p_3\cdot p_4 + p_1\cdot p_3\, p_2\cdot p_4 + p_1\cdot p_4\, p_2\cdot p_3)$

as $\frac{c_1}{M^4}\left[(s-2m^2)^2 + (t-2m^2)^2 + (u-2m^2)^2\right]$

So $\frac{d^2}{ds^2} a(s,0) = \frac{c_1}{M^4} \frac{d^2}{ds^2}\left[2(s-2m^2)^2 + (2m^2)^2\right] = \frac{4c_1}{M^4}$ as before.

However we may now have additional corrections $\sim (s-2m^2)^{2n}$ leading to $2n(2n-1)(-2m^2)^{2n-2}$
$\Rightarrow$ need $m^2 \to 0$ at end of calculation.

We may also want to apply this to EFT of compact symmetries ie non-goldstone. (The generalization of the above example to more complex G/H manifolds using a χ-Lag is straightforward, kind of).

(Actually: check this! Has it been done for anything other than $\frac{SU(2)\times SU(2)}{SU(2)}$?).

How about SMEFT? Well

① Ops of dim 6 should give $a \sim s+t+u=0$

or in any case $\frac{d^2}{ds^2}(s)=0$.

② dim 5 & 7, uh? have not thought about it?

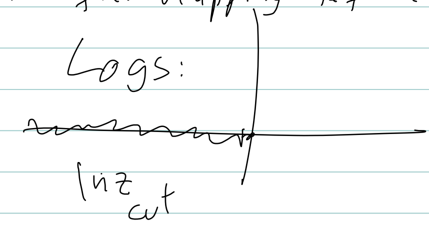
③ dim 8 ⇒ this does work for some ops that can give forward scattering

e.g. $(F_{\mu\nu}F^{\mu\nu})^2$

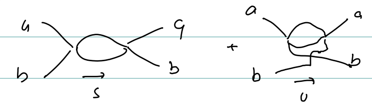
? this is $\sim(\partial A)^4$ so basically same as $(\partial\phi)^4$!

Toy example clarifying $i\epsilon$ prescription after mapping left cut to right cut.

branch cut convention $\rightarrow$ Logs:



Consider



Say 1st graph $= -\ln(4m^2-s) \to -\ln(4m^2-(s+i\epsilon)) = -\ln(\underbrace{4m^2-s}_{\text{negative}} - i\epsilon)$

$\uparrow$ below log cut $\Rightarrow -i\pi$

Hence, for $s>4m^2$ $\quad -\ln(4m^2-s) = -\left[\ln(s-4m^2) - i\pi\right]$

and now $\operatorname{Im}\left[-\ln(4m^2-s)\right] = \theta(s-4m^2)\,\pi \geq 0$

Then 2nd graph is

$$-\ln(4m^2-u) = -\left[\ln(u-4m^2) - i\pi\right] \quad \text{for } u>4m^2$$

For the dispersion relation we have the sum of these at $t=0$:

$$a(s,0) = -\ln(4m^2-s) - \ln(s)$$

And then use $s \to s+i\epsilon \Rightarrow -\ln(4m^2-s-i\epsilon) - \ln(s+i\epsilon)$

for both $s<0$ and $s>4m^2$ cuts. Now for the $s<0$ cut we

made a change of variables, $u = 4m^2-s$, which gives

(think here of u as dummy variable of integration, for which we had used "s")

$$a(u,0) = -\ln(u-i\epsilon) - \ln(4m^2-u+i\epsilon)$$

$i\epsilon$ is irrelevant here since $u>0$

opposite $i\epsilon$ to what is needed $\rightarrow$ flips Im part