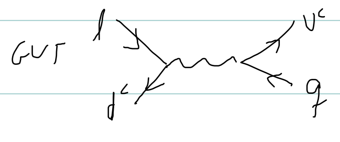
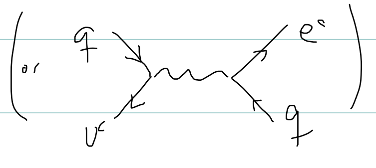
3. Beyond $\mathcal{L}_{light}$

$\tilde{\mathcal{L}}_{light}$ is the most general renormalizable lagrangian of the light fields. (It may be less general if exact symmetries of $\mathcal{L}_{full}$ forbid terms in $\mathcal{L}_{light}$). But there are processes described by $\mathcal{L}_{full}$ involving only external light fields (eg scattering of light fields) that may be entirely absent from $\tilde{\mathcal{L}}_{light}$. For example



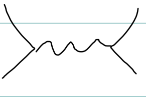


there are no terms in $\tilde{\mathcal{L}}_{light}$ one can write (consistent with gauge symmetry) that reproduce these (or that produce any $\Delta B \neq 0$). Instead we must supplement $\mathcal{L}_{eff}$ with

$$\mathcal{L}_{eff} = \tilde{\mathcal{L}}_{light} + \frac{1}{M_{GUT}} \mathcal{L}^{(5)} + \frac{1}{M_{GUT}^2} \mathcal{L}^{(6)} + \cdots$$

where $\mathcal{L}^{(n)}$ is made of operators of dimension n constructed only out of light fields and the couplings $\tilde{g}_i$.

It is easy to see how this works at tree level:



$$= i\alpha\, J_1^\mu J_2^\nu \left[-i \frac{g_{\mu\nu} - p_\mu p_\nu / M^2}{p^2 - M^2} \right] \longrightarrow \frac{i\alpha}{M} J_1^\mu J_2^\nu g_{\mu\nu} \quad \text{for } |p| \ll M$$

This is, after all, what we saw in the Introduction, the very essence of EFTs (and historically the 1st, is the case of weak interactions).

4, Beyond tree level

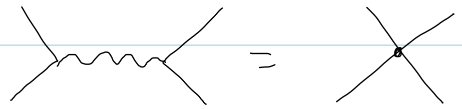
We have established (quite trivially) that in GUT for some $\Delta B \neq 0$ 4-pt functions

$$G^{(4)}(p_1 \cdots p_4) = (1)\,\tilde{G}^{(4)}(p_1 \cdots p_4) + \cdots$$

It will be more convenient to consider amputated Green's functions: $\Gamma^{(4)}(p_1,\dots,p_4) = (1)\,\tilde{\Gamma}^{(4)}(p_1 \cdots p_4)$

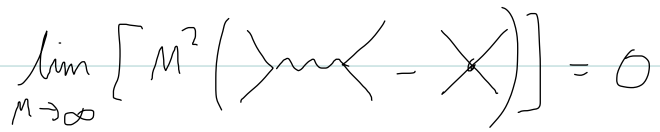
The same applies to weak interactions, if the exchanged particle is a W/Z and the external states are quarks (except t) or leptons.

Diagramatically (note: EW from here on, so we can just use $\gamma P_L \otimes \gamma P_L$ chiral structure).

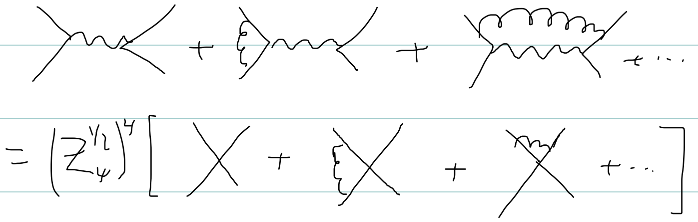


(amputated!)

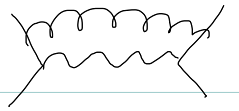
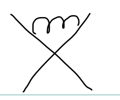
Formally,



Go to 1-loop. Stay with weak interactions as heavy, and quark + gluons as light. Then we want to show



At least for now, graphs on LHS are in 1-1 correspondance with RHS. So compare one at a time:

 $\stackrel{?}{=}$  $+ \; O\left(\frac{1}{M^3}\right)$?

↑ recall, this has explicit $\frac{1}{M^2}$

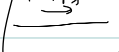
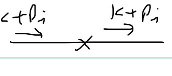
Non-sense. LHS $\sim$ (counting only) $\int d^4k \left(\frac{1}{\not k}\right)^2 \left(\frac{g_{\mu\nu}\cdots}{k^2}\right)\left(\frac{g_{\mu\nu}\cdots}{k^2-M^2}\right)$ = finite
(renormalizable gauge)

$$\text{RHS} \sim \int d^4k \left(\frac{1}{\not k}\right)^2 \left(\frac{g_{\mu\nu}\cdots}{k^2}\right)\left(\frac{1}{-M^2}\right) = \text{log divergent.}$$

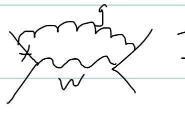
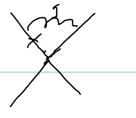
Oops! We need to renor-malize. But let's try to see if the finite part has the correct p_i dependence, while the ∞ part has "trivial" p_i dependence.

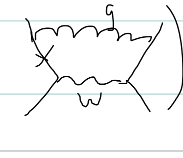
To this end take $\frac{\partial}{\partial p_i}$ on both sides of the equation. Now if k = loop mom then depending on how you route momentum through the graph, p_i (for some $i=1,\ldots,n$) will appear in at least one but possibly more lines. On any propagator $\frac{\partial}{\partial p_i}$ increases the degree of convergence:

$$\frac{\partial}{\partial p_i}\frac{1}{(k+p_i)^2-m^2} = -\frac{2k\cdot p_i}{[(k+p_i)^2-m^2]^2} \sim \frac{1}{k^3} \text{ at large } k$$

Diagramatically $\frac{\partial}{\partial p_i}\Bigg($  $\Bigg)$ =  $= \frac{1}{(k+p_i)^2-m^2}(-2k\cdot p_i)\frac{1}{(k+p_i)^2-m^2}$

Back to 4pt: take $\frac{\partial}{\partial p_i}$

 $\stackrel{?}{=}$  $+ \cdots$

To see that this works, consider $\lim\limits_{M\to\infty}\Bigg(M^2 \cdot$  $\Bigg)$

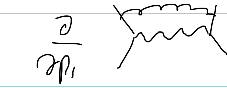
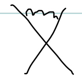
Since $\lim \int M^2(\dots)\frac{1}{k^2-M^2}$ converges uniformly and so does

$$\int \lim M^2 (\dots)\frac{1}{k^2-M^2} = \int -(\dots)$$

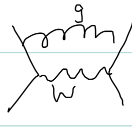
std. math nonsense ⇒ they are equal. But the RHS is just M^2  .

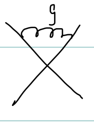
Do this for each propagator on which $\frac{\partial}{\partial p_i}$ acts, and for every $i=1,\dots,4$. (Do also $\frac{\partial}{\partial m}$ if internal light particles have masses; note m=light, M=heavy).

Note: $\frac{\partial}{\partial p_i}$ or $\frac{\partial}{\partial m}$ does not increase degree of convergence when it acts on external legs. ⇒ Consider amputated Green functions. Dealing properly with 2pt functions will result in the factor of $Z^{1/2}$ for each external leg. It is fairly superfluous, so we'll ignore from now on.

So $\frac{\partial}{\partial p_1}$  $= \frac{\partial}{\partial p_1}$  (from here on its understood this means $\lim_{M\to\infty} M^2\cdot(\ ) = \lim_{M\to\infty} M^2(\ )$).

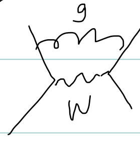
Integrating back we have

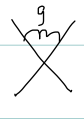


$=$


$+ f(p_2,p_3,p_4)$
↖ but not p_1

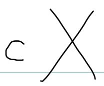
Repeating the argument for the other p_i (and m) we conclude that

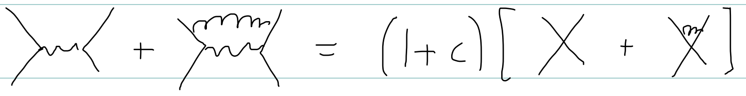


$=$


$+ \ G$

G can depend on g_s, g_w and M (namely g_2 and M_W). Moreover its infinite. Finally it must have the same chiral structure as the other terms. $\gamma^\mu(1-\gamma_5)\otimes\gamma_\mu(1-\gamma_5)$ which up to a numerical factor is 

so rewriting $C \to c$  and noting that $c \sim \mathcal{O}\left(\frac{\alpha_s}{\pi}\right)$ we have



of course this goes through when we include all 1-loop diagrams. So we obtain

$$\Gamma^{(4)}(p_1,..,p_4) = \frac{1}{M_W^2}\, G\, \tilde{\Gamma}^{(4)}_{\mathcal{O}}(p_1,..,p_4) + \ldots$$

where: Γ = amputated Green functions, renormalized

$\tilde{\Gamma}$ = idem in EFT

$\tilde{\Gamma}_{\mathcal{O}}$ = idem with a zero momentum insertion of the operator $\mathcal{O}$

G = Finite coefficient (∞-part in renormalization)

$\cdots$: higher order in $\frac{1}{M_W^2}$.

Comments

- G has an expansion, $G = 1 + c_1 \frac{\alpha_s}{\pi} + c_2\left(\frac{\alpha_s}{\pi}\right)^2 + \cdots$; it may depend on M but not on $p_1, \cdots, p_4$ nor m.
- The dependence on $p_1 \cdots p_4$ is the same $\Rightarrow$ same analytic structure (same cuts, poles, residues, what not) provided $|p_i| \ll M_W$. ("same" on full & EFTs). The two differ badly at $p_i \sim M_W$.
- The above result is summarized by stating that $\mathcal{L}_{eff} = \tilde{\mathcal{L}} + \frac{1}{M^2} G\, \mathcal{O}$
  - For many applications (eg, mixing) we need

    $$\text{amp} \propto \langle \psi_{final} | \mathcal{L}_{int} | \psi_{in} \rangle = \frac{1}{M^2} G \underbrace{\langle \psi_{final} | \mathcal{O} | \psi_{init} \rangle}_{\text{often hard to compute}}$$
  - Can be extended to all orders in perturbation theory

5. RGE improvement

So $G = G(M, g_s)$ is dimension-less → it depends on M through the ratio M/μ, μ = renormalization scale, which has been implicit.

Now

$$\mu \frac{d}{d\mu} \langle \psi_{fin} | \frac{G}{M^2} O | \psi_{init} \rangle = 0$$ because amplitudes are μ-independent

So if $\mu \frac{dO}{d\mu} = \gamma_O O \Rightarrow \mu \frac{dG}{d\mu} = -\gamma_O G$

Since $G = G(\frac{M}{\mu}, g_s)$ $\left(\frac{\partial}{\partial t} + \beta(g)\frac{\partial}{\partial g}\right) G(t,g) = -\gamma_O(g)\, G(t,g)$, $t = \ln\frac{\mu}{M}$

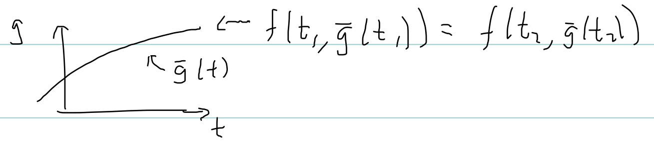
Solution: as before, let $\bar{g}$: $\frac{d\bar{g}(t)}{dt} = \beta(\bar{g}(t))$ & $\bar{g}(0,g) = g$

Before we solve for G, let's review the solution for an observable

$\frac{df}{dt} = 0$, or $\left(\frac{\partial}{\partial t} + \beta(g)\frac{\partial}{\partial g}\right) f(t,g) = 0$. We had $f(t,g) = f(0, \bar{g}(-t))$

Let's obtain this again in a manner that may shed a bit more light into the RG.

First, the Eq. $\frac{df}{dt} = 0$ means that along $\bar{g}(t)$ the function is constant:



Now, pick an arbitrary point in this t-g plane, say $\mathbb{P} = (t', g')$ (I'll drop primes at end) and set $f = f(t', g')$ for all points on the trajectory that goes through $\mathbb{P}$: $f(t', g') = f(t'+t, \bar{g}(t))$ (with $\bar{g}(0) = g'$ as before)

Then setting $t = -t'$, $f(t', g') = f(0, \bar{g}(-t'))$ (and now drop primes)

So $\frac{d}{dt}$ is just the rate of change along the trajectory

and to solve $\left(\frac{\partial}{\partial t}+\beta(g)\frac{\partial}{\partial g}\right)G(t,g) = -\gamma_0(g)\,G(t,g)$

we can evaluate this on the trajectory: $\frac{d}{dt}G = -\gamma_0(\bar{g})G$

Note that $\gamma_0(\bar{g})$ depends on t. So the equation looks like time dependent perturbation theory, $i\frac{d\psi}{dt} = H(t)\psi$. If γ_0 were a matrix, G were a vector, and $[\gamma_0(t), \gamma_0(t')] \neq 0$ then the solution would involve a t-ordered product

$$G(t) = T\, e^{-\int_0^t \gamma_0(t')dt'}\, G(0)$$

We can change variables for the integration: use $dt = \frac{d\bar{g}}{\beta(\bar{g})}$ and the fact that $\bar{g}(t)$ is monotonic

Exercise: Why is $\bar{g}(t)$ monotonic? (Assume 1 coupling only).

Then

$$G(t) = P\, e^{-\int_{\bar{g}(0)}^{\bar{g}(t)} \frac{\gamma(g')}{\beta(g')}dg'}\, G(0)$$ where the lower limit $\bar{g}(0) = g$

where $\underline{P}$ is a path-ordered (g-ordered) exponential and this can be ignored when γ_0 is not a matrix.

More on the matrix case below. But first, what does this look like at leading order:

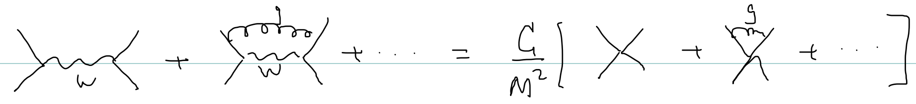
For example, at 1-loop $\gamma_0(g) = a_1 \frac{g^2}{16\pi^2}$, $\beta(g) = -b_0 \frac{g^3}{16\pi^2}$

$$\Rightarrow \quad C(t) = \exp\left(-\int_{\bar{g}(0)}^{\bar{g}(t)} \frac{c_1}{(-b_0)} \frac{dg'}{g'}\right) C(0)$$

$$= \exp\left(\frac{c_1}{b_0} \ln \frac{\bar{g}(t)}{\bar{g}(0)}\right) C(0)$$

$$= \left(\frac{\bar{\alpha}(t)}{\bar{\alpha}(0)}\right)^{\frac{c_1}{2b_0}} C(0)$$

So, as before, $C(0)$ is computed at $\mu = M$ so that there are no large logs in comparing full & effective amplitudes,



$$+ \cdots = \frac{C}{M^2}\left[\ \ + \ \ + \cdots\right]$$ (renormalized)

and therefore $C(0)$ is perturbatively computed as an expansion in powers of $\bar{\alpha}(0)$.

$$C(M) = C(0) = C_0 + \frac{\bar{\alpha}(0)}{\pi} C_1 + \cdots \qquad \text{"matching"}$$

$$C(\mu) = \left(\frac{\bar{\alpha}(\mu)}{\bar{\alpha}(M)}\right)^{\frac{c_1}{2b_0}} C(M) \qquad \text{"running"}$$

Comments:

1. From $\frac{\bar{\alpha}(0)}{\bar{\alpha}(t)} = 1 + \frac{\alpha}{2\pi} b_0 t = 1 + \frac{\alpha}{2\pi} b_0 \ln\frac{\mu}{M}$

$$\left(\frac{\bar{\alpha}(t)}{\bar{\alpha}(0)}\right)^{c_1/2b_0} = \left(1 + \frac{\alpha}{2\pi} b_0 \ln\frac{\mu}{M}\right)^{-\frac{c_1}{2b_0}} = 1 - c_1 \frac{\alpha}{4\pi} \ln\frac{\mu}{M} + \cdots + \# \left(\frac{\alpha}{4\pi} \ln\frac{\mu}{M}\right)^n \text{ terms}$$

we see again this is a sum of leading logs (LL)

The 1^{st} log term is as expected from $\left(\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial g}\right)C_1=-\gamma_0 C_1$.

One can check, explicitly, to lowest order: $\mu\frac{\partial}{\partial\mu}\left(-\frac{c_1\alpha}{4\pi}\ln\mu\right)C_1(0)=-\frac{c_1 g^2}{16\pi^2}C_1(0)$

Alternatively, ignore running of g (which is higher order) — an instructive exercise since it shows how the effect of running enters: $\mu\frac{dC_1}{d\mu}=-\gamma_0 C_1=-\frac{c_1}{16\pi^2}g^2 C_1 \Rightarrow \frac{C_1(\mu)}{C(M)}=\exp\left(-\frac{c_1}{16\pi^2}g^2\ln\frac{\mu}{M}\right)=1-\frac{c_1}{16\pi^2}g^2\ln\frac{\mu}{M}+\ldots$

2. In $\underset{\substack{\downarrow\\ \text{has } \ln\frac{\mu}{M}}}{C_1}\ \underset{\substack{\downarrow\\ \text{has } \ln\frac{\Lambda_{QCD}}{\mu}}}{\langle\psi|\mathcal{O}|\psi\rangle}$ we have separated scales (reason for EFT!)

Where do we want to deal with lage logs?

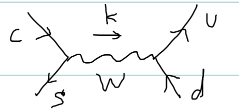
Compute C_1 even with large logs as per last page provided $\bar{\alpha}(-t)\to\bar{\alpha}(0)$ stays perturbative.

In practice use μ as low as 2 GeV (or even 1 GeV), and let non-perturbative methods deal with $\ln\frac{\Lambda}{\mu}$, a small log. (In, say, lattice QCD)

Let's do an explicit example

5.1 $\mathcal{L}_{eff}$ for $c \to s\bar{d}u$ in QED

(or $\Delta C = -\Delta U = 1$ $\Delta S = -\Delta d = -1$ $\mathcal{L}_{eff}$ in SM)

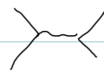
In SM 

$$= \bar{s}\gamma^\mu \frac{(1-\gamma_5)}{2} c\, \bar{u}\gamma^\nu \frac{(1-\gamma_5)}{2} d * \left(-i\frac{g_2}{\sqrt{2}}\right)^2 V_{ud}V_{cs}^*$$

$$\times -i \frac{g_{\mu\nu} - k_\mu k_\nu / M_W^2}{k^2 - M_W^2}$$ (Unitary gauge)

Here u, c, d, s are just Dirac spinors (c-numbers, 4-components labeled by momentum & spin).

Now, at $|k| << M_W$



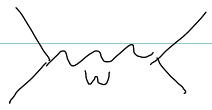
$$= -i\frac{g_2^2}{2M_W^2} V_{ud}V_{cs}^* \; \bar{s}\gamma^\mu P_L c\, \bar{u}\gamma_\mu P_L d + O\left(\frac{k^2}{M_W^2}\right)$$

So we match: $\mathcal{L}_{eff,6} = -\frac{g_2^2}{2M_W^2} V_{ud}V_{cs}^* C(M_W)\, \mathcal{O}$

("6" refers to dim $\mathcal{O}$. The dim 4 is just QCD × QED)

where $\mathcal{O}(x)$ is an operator, $\mathcal{O}(x) = \bar{s}(x)\gamma^\mu P_L c(x) \bar{u}(x) \gamma_\mu P_L d(x)$

I am pedantically indicating the u, d, s, c are now fields (operators on the Hilbert space, in fact). I won't do this, will be implicit from here on. But it's clear that the 4pt function matches that of the SM (at tree level):



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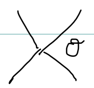


$+ O\left(\frac{k^2}{M^2}\right)$

With this we can compute, eg, $\langle \bar{K}^0 \pi^+ | i\mathcal{L}_{eff,6} | D^+ \rangle$

$$= -i\frac{g_2^2}{2M_W^2} V_{ud}V_{cs}^* C(\mu) \langle \bar{K}^0 \pi^+ | \mathcal{O} | D^0 \rangle(\mu)$$

which is μ-independent, but we only know $G(\mu)$ at $\mu = M_W$. This choice would have us compute the matrix element of $\mathcal{O}$ at $\mu = M_W$ and this would be a tough multi-scale problem.

But we know how to proceed: "run" (RGE) to $\mu \sim M_{D^0}$.

$\Rightarrow$ need to compute $\gamma_{\mathcal{O}}$.

Note that the dominant running (ie, largest logs) come from QCD, so we will should QED. But QED is simpler, so we start with that.

We compute in dimensional regularization. Use $d = 4-\epsilon$. Recall $\dim(A_\mu) = \frac{d-2}{2}$ & $\dim(\psi) = \frac{d-1}{2}$

so the bare gauge coupling has (from $\int d^dx\, g_B A^2 \partial A$

$\dim(g_B) = \frac{4-d}{2} = \epsilon/2$ (and agrees with $\int d^dx\, g_B^2 A^4$ and $g_B \bar\psi \not{A} \psi$).

So we write $g_B = \mu^{\epsilon/2} g_R Z_g(g_R)$, and I drop "R" from renormalized quantities in what follows.

— — — -

Before going any further we can already review the derivation of the formula for $\beta(g)$ in terms of Z_g:

$$\text{Since } \mu \frac{d}{d\mu} g_B = 0 \Rightarrow \tfrac{1}{2}\epsilon g Z_g + \beta(g,\epsilon)\left(Z_g + g\frac{\partial Z_g}{\partial g}\right) = 0$$

where $\beta(g,\epsilon)$ is the β function in $d = 4-\epsilon$ and we see $\beta(g,\epsilon) = -\frac{1}{2}\epsilon g + \underbrace{\beta(g,0)}_{\equiv \beta(g)}$

$$\Rightarrow \beta(g)\left(Z_g + g\frac{\partial Z_g}{\partial g}\right) - \tfrac{1}{2}\epsilon g^2 \frac{\partial Z_g}{\partial g} = 0$$

As usual $Z_g = 1 + \frac{z_g^{(1)}}{\epsilon} + \frac{z_g^{(2)}}{\epsilon^2} + \dots$ where $z_g^{(n)} = z_g^{(n)}(g)$ first comes in at n-loops. Then matching powers of ϵ we have

$$\beta(g) = \frac{1}{2} g^2 \frac{\partial z_g^{(1)}}{\partial g}$$

We won't compute this here, but we need the $-\frac{1}{2}\epsilon g$ piece ...

Similarly, $\psi_B = \mu^{-\epsilon/2} Z_\psi^{1/2} \psi$ and $A_{\mu B} = \mu^{-\epsilon/2} Z_A^{1/2} A_\mu$

Now the factor of $\mu^{-\epsilon/2}$ is fairly inconsequential (it shifts $\gamma_{\psi,A}$ by $\frac{1}{2}\epsilon$). So ignoring it

If $\mu\frac{d\psi}{d\mu} = \gamma_\psi \psi \Rightarrow 0 = \beta(g,\epsilon)\frac{1}{2}Z^{-1/2}\frac{\partial Z_\psi}{\partial g} + Z_\psi^{1/2}\gamma_\psi$

So writing $Z_\psi = 1 + \frac{z_\psi^{(1)}}{\epsilon} + \dots$ and using $\beta(g,\epsilon) = -\frac{1}{2}\epsilon g + \beta(g)$

$$\Rightarrow \gamma_\psi = \frac{1}{4} g \frac{\partial z_\psi^{(1)}}{\partial g}$$

Finally, and similarly $\mathcal{O}_R = Z_\mathcal{O} \mathcal{O}_B \Rightarrow \beta(g,\epsilon)\frac{\partial Z_\mathcal{O}}{\partial g} = Z_\mathcal{O}\gamma_\mathcal{O}$

$Z_\mathcal{O} = 1 + \frac{z_\mathcal{O}^{(1)}}{\epsilon} + \dots \Rightarrow \gamma_\mathcal{O} = -\frac{1}{2} g \frac{\partial z_\mathcal{O}^{(1)}}{\partial g}$

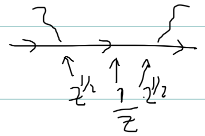
— — —

The precise meaning of "$\mu\frac{d\mathcal{O}}{d\mu}$" is from Green functions:

generically $G_\mathcal{O}^{(n)}(x_1,\dots,x_n;y) = \langle 0 | T\big(\mathcal{O}(y)\,\phi(x_1)\cdots\phi(x_n)\big)|0\rangle$

and $G_{\mathcal{O}B}^{(n)} = Z_\mathcal{O}^{-1} Z^{n/2} G_{\mathcal{O}R}^{(n)}$

(Note: if you get confused with $Z^{\pm 1/2}$ in Green's functions compute with $Z\bar{\Psi} i \partial \Psi$ or $Z(\partial\phi)^2$ and $(Z_\psi^{1/2}\bar{\Psi})(Z_A^{1/2} A\!\!\!/)(Z_\psi^{1/2}\Psi)$ int's.
Propagator $\frac{1}{Z}\frac{i}{p\!\!\!/}$, cancels Z in all internal lines



For amputated graphs, $\Gamma^{(n)}$, this gives a $Z^{1/2}$ for each external amputated leg. This means that if one computes with bare $\mathcal{L}$ $\Gamma^{(n)}(g_B)$ then one can take this function, replace $g_B = Z_g g_R$ and multiply by $Z^{n/2}$ and that would be the same as obtained from the renormalized $\mathcal{L}$:

$$\Gamma_R^{(n)}(g_R) = Z^{n/2}\,\Gamma_B^{(n)}(Z_g g_R)$$

$G^{(n)}$: Since each external line has a full propagator attached to each leg in $\Gamma^{(n)}$, and $G^{(2)} = 1/\Gamma^{(2)}$ then

$$(i)\quad G_R^{(2)}(g_R) = 1/\Gamma_R^{(2)} = Z^{-1}/\Gamma_B^{(2)}(Z_g g_R) = Z^{-1} G_B^{(2)}(Z_g g_R)$$

$$(ii)\quad G_R^{(n)}(g_R) = [G_R^{(2)}]^n\,\Gamma_R^{(n)} = Z^{-n/2}\,G_B^{(n)}(Z_g g_R)$$

With an insertion of an operator $\Gamma_{\mathcal{O}R}^{(n)}(g_R) = Z^{n/2} Z_{\mathcal{O}}\,\Gamma_{\mathcal{O}B}^{(n)}(Z_g g_R)$

The inverse in $Z_{\mathcal{O}}^{-1}$ is completely arbitrary and I chose $\mathcal{O}_R = Z_{\mathcal{O}}\,\mathcal{O}_B$.

Incidentally $\mu\frac{d}{d\mu}\Gamma_R^{(n)} = \mu\frac{d}{d\mu}Z^{n/2}\,\Gamma_B^{(n)} = \frac{n}{2}\gamma_\psi \Gamma_R$, $\mu\frac{d}{d\mu}G^{(n)} = -\frac{n}{2}\gamma_\psi\, G^{(n)}$

and $\mu\frac{d}{d\mu}\Gamma_{\mathcal{O}R}^{(n)} = \mu\frac{d}{d\mu}(Z^{n/2} Z_{\mathcal{O}}\,\Gamma_{\mathcal{O}B}) = \left(\frac{n}{2}\gamma_\psi + \gamma_{\mathcal{O}}\right)\Gamma_{\mathcal{O}R}^{(n)}$).

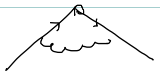
NOT FOR CLASS

Want to quickly check on Z_O.

For amputated

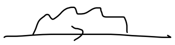
$$\Gamma_O^R = Z^{n/2} Z_O \Gamma^B$$

Take O: $J^\mu = \bar\Psi \gamma^\mu \psi$



$$= \int \frac{d^n k}{(2\pi)^n} ie\gamma^\lambda i\frac{\not k + m}{k^2-m^2} \gamma^\mu i\frac{\not k + m}{k^2-m^2} ie\gamma_\lambda \left(-i\frac{1}{k^2}\right)$$

$$= -ie^2 (-2)^2 \frac{1}{4}\gamma^\mu \int \frac{d^n k}{(2\pi)^n} \frac{1}{(k^2-m^2)^2} = -ie^2\gamma^\mu \frac{i}{16\pi^2}\Gamma(2-\tfrac{n}{2}) = \frac{e^2}{16\pi^2}\frac{2}{\epsilon}\gamma^\mu$$



$$= \int \frac{d^n k}{(2\pi)^n} \frac{-i}{k^2} \, -ie\gamma^\lambda i\frac{\not k + \not p}{(k+p)^2} \, -ie\gamma_\lambda = -e^2(-2)\int_0^1 dx \int \frac{d^n k}{(2\pi)^n} \frac{(1-x)\not p}{(k^2+x(1-x)p^2)^2}$$

$$= 2e^2 \not p \frac{i}{16\pi^2}\frac{2}{\epsilon}\frac{1}{2}$$

This is $i\Sigma$, so $\Sigma_k = \frac{e^2}{16\pi^2}\frac{2}{\epsilon}$, $Z(1+\Sigma_k) = \text{finite} \Rightarrow Z = 1 - \frac{e^2}{16\pi^2}\frac{2}{\epsilon}$

$\Gamma^R = \text{finite} = Z Z_O \Gamma^B = Z_O\left(1 - \frac{e^2}{16\pi^2}\frac{2}{\epsilon}\right)\left(1 + \frac{e^2}{16\pi^2}\frac{2}{\epsilon}\right)\gamma^\mu \Rightarrow Z_O - 1 = 0$. ✓

For a conserved current, $\gamma_J = 0$. ✓✓

(Incidentally $Z_\psi = 1 - \frac{e^2}{16\pi^2}\frac{2}{\epsilon} \Rightarrow \gamma_\psi = \frac{1}{4}e\frac{\partial}{\partial e}\left(-\frac{e^2}{8\pi^2}\right) = -\frac{e^2}{32\pi^2}$).

1-loop integral formula for dim reg

$$I_{p,r} = \int \frac{d^dk}{(2\pi)^d} \frac{(k^2)^p}{(k^2-M^2)^r} = i(-1)^{p-r} \frac{\pi^{d/2}/\Gamma(d/2)}{(2\pi)^d} \int_0^\infty (k^2)^{\frac{d-2}{2}} dk^2 \frac{(k^2)^p}{(k^2+m^2)^r}$$

$$= i(-1)^{p-r} \frac{1}{(4\pi)^{d/2}} \frac{(M^2)^{\frac{d}{2}+p-r}}{\Gamma(d/2)} \int_0^\infty dx \frac{x^{p+\frac{d}{2}-1}}{(x+1)^r}$$

Let $y = \frac{1}{x+1}$ $\quad dy = -\frac{1}{(x+1)^2}dx = -y^2 dx \quad dx = -\frac{dy}{y^2}$

$x = \frac{1}{y} - 1 = \frac{1-y}{y}$

$$\int_0^\infty dx \frac{x^{p+\frac{d}{2}-1}}{(x+1)^r} = \int_0^1 \frac{dy}{y^2} \left(\frac{1-y}{y}\right)^{p+\frac{d}{2}-1} y^r = \int_0^1 dy\, y^{r-p-\frac{d}{2}-1} (1-y)^{p+\frac{d}{2}-1} = B(r-p-\tfrac{d}{2}, p+\tfrac{d}{2})$$

where $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$

$$\boxed{I_{p,r} = \frac{i}{(4\pi)^{d/2}} (-1)^{p-r} (M^2)^{\frac{d}{2}+p-r} \frac{\Gamma(r-p-\frac{d}{2})\Gamma(p+\frac{d}{2})}{\Gamma(r)\,\Gamma(d/2)}}$$

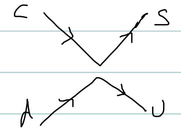
Also useful $\quad \frac{1}{ab} = \int_0^1 dx \frac{1}{(ax+b(1-x))^2}$

and derivatives, $\frac{\partial}{\partial a} \rightarrow \quad \frac{1}{a^2b} = 2\int_0^1 dx \frac{x}{(ax+b(1-x))^3}$ etc.

We'll use extensively

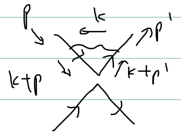
$$I_{0,2} = \frac{i}{(4\pi)^{d/2}} (M^2)^{\frac{d}{2}-2} \Gamma(2-d/2) = \frac{i}{16\pi^2} \frac{2}{\epsilon}\left(1+\mathcal{O}(\epsilon)\right)$$

Compute $\gamma_\mathcal{O}$ for $\mathcal{O} = \bar{s}\gamma^\mu P_L c\, \bar{u}\gamma_\mu P_L d$ in QED



I will (i) omit labels "c", "d", ... below
(ii) work with arbitrary charges Q_c, Q_s, etc.
(helps check calculation)

$$\text{Feynman rule} = \gamma^\mu P_L \otimes \gamma_\mu P_L$$



$$= \int \frac{d^dk}{(2\pi)^d} \left[(-ieQ_s\gamma^\lambda) \frac{i}{\not k+\not p' - m_s} \gamma^\mu P_L \frac{i}{\not k + \not p - m_c} (-ieQ_c\gamma^\rho) \right] \otimes \gamma_\mu P_L \left[-i\frac{\eta_{\lambda\rho}}{k^2} \right]$$

I have chosen Feynman gauge. Landau gauge is also a convenient choice because at 1-loop the quark self-energy is finite (so no need to compute [illegible] in that gauge).

Since we are only interested in UV log divergent part, that gives a $\frac{1}{\epsilon}$, we can set $p' = p = 0$ (ie, expand in powers of p/k and p'/k, but then the p-dependent terms are finite). We cannot also ignore masses since we would get IR divergences which will show up as additional $1/\epsilon$'s and we don't know how much UV vs IR in $\frac{1}{\epsilon_{UV}} + \frac{1}{\epsilon_{IR}}$, so to speak.

But we can set $m_c = m_s$, since the $\frac{1}{\epsilon}$ term is m_q independent.

Alternatively

$$\frac{1}{k^2-m_c^2} = \frac{1}{k^2-M^2} + \left(\frac{1}{k^2-m_c^2} - \frac{1}{k^2-M^2} \right) = \frac{1}{k^2-M^2} + \frac{m_c^2-M^2}{(k^2-m_c^2)(k^2-M^2)}$$

and the last term gives a finite contribution to the integral. So

$$= -ie^2Q_sQ_c \int \frac{d^dk}{(2\pi)^d} \frac{\gamma^\lambda \not k \gamma^\mu P_L \not k \gamma_\lambda \otimes \gamma_\mu P_L}{(k^2-M^2)^2 k^2} + \text{finite}$$

Use $\int d^dk\, f(k^2)\, k^\alpha k^\beta = \int d^dk\, f(k^2)\, \frac{1}{d} k^2 \eta^{\alpha\beta}$

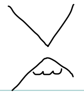
and $\frac{1}{d}\gamma^\lambda\gamma^\alpha\gamma^\mu P_L \gamma_\alpha \gamma_\lambda = \frac{1}{d}\gamma^\lambda\gamma^\alpha\gamma^\mu\gamma_\alpha\gamma_\lambda P_L = \frac{1}{d}(2-d)^2\gamma^\mu P_L \to \gamma^\mu P_L \quad (+\mathcal{O}(\epsilon))$.

(Note that I used $P_L\gamma^\mu = \gamma^\mu P_R$, "naive γ_5": no subtleties at 1-loop)

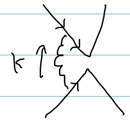
$$= -ie^2Q_sQ_c\gamma^\mu P_L \otimes \gamma_\mu P_L \underbrace{\int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2-M^2)^2}}_{\frac{i}{16\pi^2}\frac{2}{\epsilon}} = \frac{1}{\epsilon}\left[2\frac{e^2}{16\pi^2}Q_sQ_c \right] \gamma^\mu P_L \otimes \gamma_\mu P_L$$

↑ this is the only factor that requires calculation

☺



same with $Q_sQ_c \to Q_dQ_u$ $\quad = \frac{1}{\epsilon}\left[2\frac{e^2}{16\pi^2}Q_uQ_d\right]\gamma^\mu P_L \otimes \gamma_\mu P_L$



$$= (-ie)^2 Q_c Q_d \int \frac{d^dk}{(2\pi)^d} \frac{\gamma^\mu P_L\, i\not{k}\,\gamma^\lambda \otimes \gamma_\mu P_L(-i\not{k})\gamma_\lambda}{(k^2-M^2)^2}\left(-\frac{i}{k^2}\right)$$

$$= ie^2 Q_c Q_d \frac{1}{d}\gamma^\mu P_L \gamma^\alpha\gamma^\lambda \otimes \gamma_\mu P_L \gamma_\alpha \gamma_\lambda \left[\frac{2}{\epsilon}\frac{i}{16\pi^2}\right]$$

The γ-matrix algebra can be done in $d=4$. Using

$$\gamma^\mu\gamma^\alpha\gamma^\lambda = \gamma^\mu\eta^{\alpha\lambda} - \gamma^\alpha\eta^{\mu\lambda} + \gamma^\lambda\eta^{\mu\alpha} + i\epsilon^{\mu\alpha\lambda\rho}\gamma_\rho\gamma_5\,, \qquad \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

$$\epsilon^{0123} = -\epsilon_{0123} = +1$$

I get $\gamma^\mu\gamma^\alpha\gamma^\lambda P_L \otimes \gamma_\mu\gamma_\alpha\gamma_\lambda P_L = 16\,\gamma^\mu P_L \otimes \gamma_\mu P_L$, so

$$= \frac{1}{\epsilon}\left[-8\frac{e^2}{16\pi^2}Q_cQ_d\right]\gamma^\mu P_L \otimes \gamma_\mu P_L$$

Likewise

$$= \frac{1}{\epsilon}\left[-8\frac{e^2}{16\pi^2}Q_sQ_u\right]\gamma^\mu P_L \otimes \gamma_\mu P_L$$

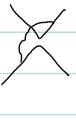
Finally

$$= (-ie)^2 Q_c Q_u \int \frac{d^dk}{(2\pi)^d} \frac{\gamma^\mu P_L\, i\not{k}\,\gamma^\lambda \otimes \gamma_\lambda\, i\not{k}\,\gamma_\mu P_L}{(k^2-M^2)^2}\left(-i\frac{1}{k^2}\right)$$

$$= -ie^2 Q_c Q_u \frac{1}{d}\gamma^\mu P_L \gamma^\alpha\gamma^\lambda \otimes \gamma_\lambda\gamma_\alpha\gamma_\mu P_L \left[\frac{2}{\epsilon}\frac{i}{16\pi^2}\right]$$

and now $\gamma^\mu\gamma^\alpha\gamma^\lambda P_L \otimes \gamma_\lambda\gamma_\alpha\gamma_\mu P_L = 4\,\gamma^\mu P_L \otimes \gamma_\mu P_L$

$$= \frac{1}{\epsilon}\left[2\frac{e^2}{16\pi^2}Q_cQ_u\right]\gamma^\mu P_L \otimes \gamma_\mu P_L$$



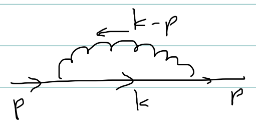
$$= \frac{1}{\epsilon}\left[2\frac{e^2}{16\pi^2}Q_sQ_d\right]\gamma^\mu P_L \otimes \gamma_\mu P_L$$

Note that if we ignore color, $\bar{s}\gamma^\mu P_L c\,\bar{u}\gamma_\mu P_L d = \bar{s}\gamma^\mu P_L d\,\bar{u}\gamma^\mu P_L c$ so the result should be invariant under the change of labels $d \leftrightarrow c$ or $u \leftrightarrow s$

$$\text{SUM} = \frac{1}{\epsilon}\frac{2e^2}{16\pi^2}\left[Q_sQ_c + Q_dQ_u + Q_sQ_d + Q_cQ_u - 4(Q_sQ_u + Q_cQ_d)\right]\gamma^\mu P_L \otimes \gamma_\mu P_L$$

$$\equiv \frac{1}{\epsilon}G\,\gamma^\mu P_L \otimes \gamma_\mu P_L$$ (G stands for sum of Graphs).

We also need



$$= \int \frac{d^dk}{(2\pi)^d} (-ieQ)\gamma^\lambda \frac{i\not{k}}{k^2}(-ieQ\gamma_\lambda)\left(\frac{-i}{(k-p)^2}\right)$$

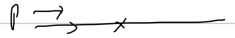
$$= 2e^2Q^2 \int \frac{d^dk}{(2\pi)^d} \frac{\not{k}}{k^2(k-p)^2}$$

$$= 2e^2Q^2 \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{\not{k}}{(k^2 - 2k\cdot px + xp^2)^2}$$

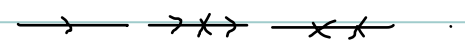
$$= 2e^2Q^2\not{p} \int_0^1 dx\, x \underbrace{\int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2 + x(1-x)p^2)^2}}_{\frac{2}{\epsilon}\frac{i}{16\pi^2}}$$

$$= \frac{1}{\epsilon}\left[2\frac{e^2}{16\pi^2}Q^2\right] i\not{p}$$

$\mathcal{L} \supset Z\bar{\Psi} i\not{\partial}\psi = \bar{\Psi}i\not{\partial}\psi + (Z_\psi - 1)\bar{\psi}i\not{\partial}\psi$ insertion of $i\mathcal{L}$



$= (Z_\psi - 1)\, i\not{p}$ (Quick check: resumming



gives $\frac{i}{Z\not{p}}$ ✓).

$$\Rightarrow \quad Z_\psi - 1 + \frac{1}{\epsilon}\left[2\frac{e^2}{16\pi^2}Q^2\right] = 0 \quad (\text{in MS})$$

$$Z_\psi = 1 + \frac{Z_\psi^{(1)}}{\epsilon} \qquad Z_\psi^{(1)} = -2\frac{e^2}{16\pi^2}Q^2$$

Determine Z_θ: $\Gamma_R = (Z^{1/2})^4 Z_\theta \Gamma_B$ = finite

$$= (Z^{1/2})^4 Z_\theta\, \gamma^\mu P_L \otimes \gamma_\mu P_L \left(1 + \frac{1}{\epsilon}G\right)$$

Use $Z_\theta = 1 + \frac{Z_\theta^{(1)}}{\epsilon}$

$$1 + \frac{1}{\epsilon}\left(\frac{1}{2}\sum_1^4 Z_\psi^{(1)} + Z_\theta + G\right) = \text{finite}$$

$$Z_\theta^{(1)} = -G - \frac{1}{2}\sum_1^4 Z_\psi^{(1)}$$

$$= -\frac{2e^2}{16\pi^2}\left[Q_sQ_c + Q_dQ_u + Q_sQ_d + Q_cQ_u - 4(Q_sQ_u + Q_cQ_d) - \frac{1}{2}(Q_s^2 + Q_c^2 + Q_u^2 + Q_d^2)\right]$$

and $\gamma_\theta = -\frac{1}{2}e\frac{\partial Z_\theta^{(1)}}{\partial e} = \frac{c_1 e^2}{16\pi^2}$ where $c_1 = 2\left[\quad\downarrow\quad\right] = 2\left(-6\left(\frac{2}{3}\right)\left(-\frac{1}{3}\right)\right) = \frac{8}{3}$

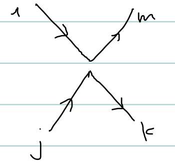
One check: if $Q_c = Q_s$ & $Q_d = Q_u = 0$ should have $\gamma_\theta = 0$ by current conservation.

5.2 Operator Mixing (and back to QCD).

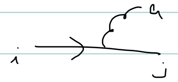
Consider the renormalization of the same operator but now using QCD rather than QED (and still only 1-loop).

The graphs are the same, but there are important differences: the gluon-quark-quark vertex has a matrix structure and we should keep track of color of external quarks (the Green function

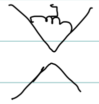
$\langle 0 | T\, \mathcal{O}(0)\, \bar{c}^i(x)\, \bar{d}^j(y)\, u^k(z)\, s^m(w) | 0\rangle$ is labeled by the colors of the 'external' quarks, i, j, k, m. So our Feynman diagrams have:



$\propto \delta^{im}\, \delta_{jk}$ which I denote as $\mathbb{1} \otho \mathbb{1}$

and 

$\propto T^a_{ji}$

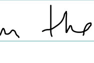
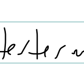
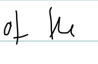
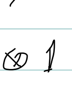
So now, in  instead of $Q_c Q_s$ we have $T^a T^a \otimes \mathbb{1}$

$= C_2(R_f)\ \mathbb{1} \otimes \mathbb{1}$

Where the Casimir for rep R is $T^a T^a = C^2(R)\ \mathbb{1}$ and is related to $T_2(R)$ of $\mathrm{Tr}\, T^a T^b = T_2(R)\, \delta^{ab}$ fame by $\mathrm{Tr}(T^a T^a) = C(R)\, \dim(R) = T(R)\, \dim(\mathrm{Adj})$

so, eg, for fundamentl of $SU(N)$, $C(R) = \frac{1}{2N}(N^2-1)$, or $\frac{4}{3}$ for $SU(3)$.

Likewise 

But for  we have $\propto T^a \otimes T^a$; same for , , 

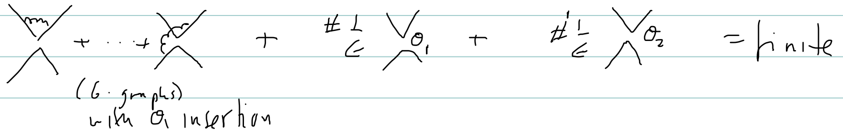
The $\frac{1}{\epsilon}$ from these cannot be subtracted by a counterterm of the $\mathbb{1} \otimes \mathbb{1}$ form.

We need to introduce a second operator, $\bar{c}\, T^a \gamma^\mu P_L\, s\ \bar{u}\, T^a \gamma_\mu P_L\, d$, as counterterm.

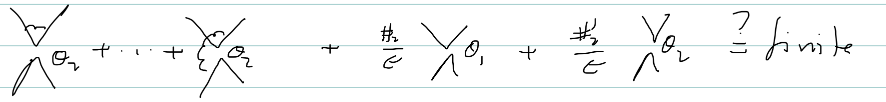
So Let $\mathcal{O}_1 = \bar{c}\gamma^\mu P_L s \, \bar{u}\gamma_\mu P_L d$

$\mathcal{O}_2 = \bar{c}T^a\gamma^\mu P_L s \, \bar{u}T^a\gamma_\mu P_L d$

What we are saying is that



Then, you may ask, do I need additional operators to subtract insertions of $\mathcal{O}_2$?



The answer is yes. We'll verify this explicitly but first we give a general argument. This type of argument is very useful in characterizing EFT's — not just in understanding renormalization and RG.

Claim: Only operators with the same quantum numbers are required for their renormalization.

We say that the set of operator "closes" under renormalization

You may ask: what does it matter whether I have counterterms for $\mathcal{O}_2$? After all I need to consider only $\mathcal{O}_1$ since this is what I get from matching $\mathcal{L}_{full}$ to $\mathcal{L}_{eff}$.

Not quite: if $\mathcal{L}_{eff} \supset \frac{1}{M^2}\left(C_1(\mu)\,\mathcal{O}_1(\mu) + C_2(\mu)\mathcal{O}_2(\mu)\right)$ Then the effect of the off-diagonal renormalization is that even if $C_2(M) = 0$ (and $C_1(M) \neq 0$, of course) at the matching scale, $\mu = M$, one gets $C_2(\mu) \neq 0$ at $\mu < M$. We will show this shortly.

Back to the claim: if the regularization procedure respects the symmetries of the theory, then covariance under all symmetries is explicit in calculations of Feynman diagrams. (This is basic QFT material, so I won't review it).

In our case massless QCD has seprate $U(1)$ symmetries for u_L, c_L, d_L and s_L. (It's anomalous, but only broken by instantons → irrelevant)
So the operator must contain $\bar{u}_L, \bar{s}_L, c_L$ and d_L

$\mathcal{O}_1$ is a Lorentz and color scalar. So should be the operators in the closed set.

In dim reg without masses (mass independent subtraction scheme) only operators with same mass dimension as $\mathcal{O}$ ($\dim\mathcal{O}=6$) are needed.

So the ops we are looking for are made out of exactly the 4 fields, and to make Lorentz scalars

$$\bar{s}^i_L \gamma^\mu c^j_L \, \bar{u}^k_L \gamma_\mu d^m_L \quad \text{and} \quad \bar{s}^i_L \gamma^\mu d^m_L \, \bar{u}^k_L \gamma^\mu c^j_L$$

where i,j,k,m are $SU(3)$ indices. Now the two above are equal (Fierz rearrange!)
Finally the ops must be color singlets. How many independent singlets are in this op? The 4 fields are two 3's & two $\bar{3}$'s, and we need to find how many ways are there to combine them into singlet.

Well

$$3\times\bar{3} = 1\oplus 8$$
$$1\times 1 = 1$$
$$8\times 8 = 1 + 8 + 8 + 10 + \overline{10} + 27$$

So $(3\times\bar{3})\times(3\times\bar{3}) = (1\oplus 8)\otimes(1\oplus 8) = 1\oplus 1\oplus$ non-singlets

$\Rightarrow$ 2 invariants, one in 1×1 and the other in 8×8

$\Rightarrow \underline{\mathcal{O}_1 \text{ and } \mathcal{O}_2} \text{ are it}$!

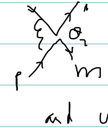
To verify this directly, use the color Fierz rearrangement formula

$$T^a_{ij} T^a_{mn} = \frac{1}{2}\left(\delta_{in}\delta_{mj} - \frac{1}{3}\delta_{ij}\delta_{mn}\right)$$

Then

$$\sim (T^b T^a T^b)_{in} \otimes T^b = \frac{1}{2}\left(\delta_{in}\delta_{mj} - \frac{1}{3}\delta_{ij}\delta_{mn}\right) T^a_{jm}\otimes T^a = -\frac{1}{6} T^a_{in}\otimes T^a$$

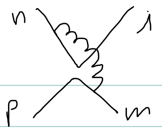
(that this is $\sim T^a\otimes T^a$ was obvious)



$$(T^b T^a)_{in} (T^b T^a)_{mp} = (T^b_{ij} T^a_{jn})(T^b_{mk} T^a_{kp}) = \frac{1}{2}\left(\delta_{ik}\delta_{mj} - \frac{1}{3}\delta_{ij}\delta_{mk}\right) T^a_{jn} T^a_{kp} = \frac{1}{2}\left(T^a_{mn}T^a_{ip} - \frac{1}{3}T^a_{in}T^a_{mp}\right)$$

and use $T^a_{mn}T^a_{ip} = \frac{1}{2}\delta_{mp}\delta_{in} - \frac{1}{6}\delta_{mn}\delta_{ip} = \frac{1}{2}\delta_{mp}\delta_{in} - \frac{1}{6}\left(2T^a_{mp}T^a_{in} + \frac{1}{3}\delta_{mp}\delta_{in}\right) = \frac{4}{9}1\otimes 1 - \frac{1}{3}T^a\times T^a$

so $T^a T^b \otimes T^a T^b = \frac{1}{2}\left(\frac{4}{9}1\otimes 1 - \frac{1}{3}T^a\times T^a\right) - \frac{1}{6}T^a\otimes T^a = \frac{2}{9}1\otimes 1 - \frac{1}{3}T^a\otimes T^a$



$$\sim (T^a T^b)_{in} (T^b T^a)_{mp} = T^a_{ij} T^b_{jn} T^b_{mk} T^a_{kp}$$

$$= T^a_{ij} T^a_{kp} \frac{1}{2} \left(\delta_{jk} \delta_{mn} - \frac{1}{3} \delta_{jn} \delta_{mk} \right)$$

$$= \frac{1}{2} (T^a T^a)_{ip} \delta_{mn} - \frac{1}{6} T^a \otimes T^a$$

$$= \frac{2}{3} \delta_{ip} \delta_{mn} - \frac{1}{6} T^a \times T^a$$

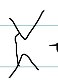
$$\left(2 T^a_{in} T^a_{mp} = \delta_{ip} \delta_{mn} - \frac{1}{3} \delta_{in} \delta_{mp} \Rightarrow \delta_{ip} \delta_{mn} = \frac{1}{3} 1 \otimes 1 + 2 T^a \otimes T^a \right)$$

$$= \frac{2}{3} \left(\frac{1}{3} 1 \times 1 + 2 T^a \times T^a \right) - \frac{1}{6} T^a \times T^a$$

$$= \frac{2}{9} 1 \otimes 1 - \frac{7}{6} T^a \otimes T^a$$

We can complete the calculation by reading off the rest of the integral from the QED case;

 +  =  +  $= \frac{1}{\epsilon} 4 \frac{g^2}{16\pi^2}$

 +  $= \frac{1}{\epsilon} (-16) \frac{g^2}{16\pi^2}$

and combining with color factors for 1×1 & $T \times T$ separately:

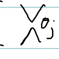
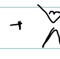
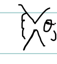
$$\mathcal{O}_1 : \frac{1}{\epsilon} 4 \frac{g^2}{16\pi^2} \left[(1 \otimes 1) \left(\frac{4}{3} \right) + (T^a \otimes T^a) \left(1 (1) - 4 (1) \right) \right] = \frac{1}{\epsilon} \frac{g^2}{16\pi^2} \left[\frac{16}{3} 1 \otimes 1 - 12 T^a \otimes T^a \right]$$

$$\mathcal{O}_2 = \frac{1}{\epsilon} 4 \frac{g^2}{16\pi^2} \left[1 \otimes 1 \left(1 \cdot (0) - 4 \left(\frac{2}{9} \right) + 1 \left(\frac{2}{9} \right) \right) + T^a \otimes T^a \left[1 \cdot \left(-\frac{1}{6} \right) - 4 \left(-\frac{1}{3} \right) + 1 \left(-\frac{7}{6} \right) \right] \right]$$

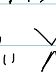
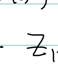
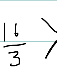
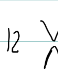
$$= \frac{1}{\epsilon} \frac{g^2}{16\pi^2} \left[-\frac{8}{3} 1 \otimes 1 + 0 \, T^a \otimes T^a \right]$$

Then to subtract these divergences:

$$\Gamma^R_{O_i}(g_R) = Z_{ij} (Z^{1/2})^4 \, \Gamma^B_{O_j} (Z_g g_R)$$

$= Z_{ij} (Z^{1/2})^4 \Big[$  ^{O_j} +  ^{O_j} +  $^{O_j} + \cdots \Big]$

To get an idea of how this works, to 1-loop,

$\Gamma^R_{O_1} = (Z^{1/2})^4 \Big[Z_{11}$  $^{O_1} + Z_{12}$  $^{O_2} + \frac{1}{\epsilon} \frac{g^2}{16\pi^2} \frac{16}{3}$  $^{O_1} - \frac{1}{\epsilon} \frac{g^2}{16\pi^2} 12$  $^{O_2} \Big]$

so that $Z_{12} - \frac{1}{\epsilon} \frac{12 g^2}{16\pi^2} = 0$ etc.

Exercise: (i) Compute wave-function renormalization constant Z_ψ

(ii) Use this to complete the calculation of the 2x2 matrix of numbers $Z^{(1)}_{ij}$ in

$$Z_{ij} = \delta_{ij} + \frac{Z^{(1)}_{ij}}{\epsilon} + \cdots$$

The RGE is now
$$\mu \frac{d}{d\mu} \Gamma^{(n)R}_{\theta_i} = \mu \frac{d}{d\mu}\left((Z^{1/2}_\psi)^n Z_{ij} \Gamma^B_{\theta_j} \right)$$

$$= \left[\frac{n}{2}\gamma_\psi \delta_{ij} + (\gamma_\theta)_{ij} \right] \Gamma^{(n)R}_{\theta_j}$$

or
$$\left[\mu \frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} - \frac{n}{2}\gamma_\psi \right] \Gamma^{(n)R}_{\theta_i} = (\gamma_\theta)_{ij} \Gamma^{(n)R}_{\theta_j}$$

Here $\mu \frac{dZ_{ij}}{d\mu} \Gamma^B_{\theta_j} = (\gamma_\theta)_{ik} Z_{kj} \Gamma^B_{\theta_j}$, in matrix notation $\gamma_\theta = \mu \frac{dZ}{d\mu} Z^{-1}$

So now $\mathcal{L}_{eff} = \frac{1}{M^2} C_i(\mu) \theta_i(\mu)$ and as before $\mu \frac{d\mathcal{L}_{eff}}{d\mu} = 0$

$$\Rightarrow \quad \mu \frac{dC_i}{d\mu} \theta_i + C_i (\gamma_\theta)_{ij} \theta_j = 0$$

$$\mu \frac{dC_i}{d\mu} = -C_j (\gamma_\theta)_{ji} = -(\gamma_\theta^T)_{ij} C_j$$

$$\text{or } \mu \frac{d\vec{C}}{d\mu} = -\gamma_\theta^T \vec{C}$$

As before
$$\vec{C}(\mu) = P \exp\left(-\int_{\bar{g}(M)}^{\bar{g}(\mu)} \frac{dg'}{\beta(g')} \gamma_\theta^T(g') \right) \vec{C}(M)$$

As a result, if $C_2(M) = 0$ but $C_1(M) \neq 0$, then for $\mu \neq M$ $C_2(\mu) \neq 0$

At 1-loop $[\gamma_\theta(g), \gamma_\theta(g')] = 0$ because $\gamma_\theta(g) = \frac{g^2}{16\pi^2} c_1$ (c_1 a matrix of numbers, g-independent).

$\Rightarrow$ at 1-loop no need for P-ordering. Moreover, an arbitrary real NxN matrix is diagonalized by $U c_1 U^{-1} = c_1^{diag}$ with U an invertible NxN matrix, and $U^{T^{-1}} c_1^T U^T = c_1^{diag}$.

So, at 1-loop
$$\vec{C}(\mu) = U^T \underbrace{\left[U^{T^{-1}} e^{-\int_{\bar{g}(M)}^{\bar{g}(\mu)} \gamma_\theta^T(g') \frac{dg'}{\beta(g')}} U^T \right]}_{\hookrightarrow \text{ a diagonal matrix } = \left(\frac{\alpha(\mu)}{\alpha(M)} \right)^{\frac{c_1^{diag}}{2b_0}}} U^{T^{-1}} \vec{C}(M)$$

5.3 More operator mixing and the use of Equations of Motion

The set of operators that close under renormalization can get large. In principle it is infinite, but if we consider (as we have) only mass independent subtraction schemes then the number is finite: an operator of mass dimension $d_{\mathcal{O}}$ only mixes with operators of the same mass dimension $d_{\mathcal{O}}$.

As we have seen one can further constrain this set by use of symmetries. Moreover, the symmetry need not be exact. Define a softly broken symmetry as one that is restored (becomes exact) as some parameters with positive mass dimension (like masses or cubic scalar couplings) are set to zero. The violation of symmetry must be proportional to these parameters so an operator of dimension $d_{\mathcal{O}}$ woul require counterterms that are $\sim m^n \mathcal{O}'$ with $\dim(\mathcal{O}') = d_{\mathcal{O}} - n < d_{\mathcal{O}}$; but that won't happen in a mass independent scheme. Stated differently, in a mass independent scheme one may set these parameters $m \to 0$ and restore symmetries.

If under a symmetry group G the operator $\mathcal{O}$ transforms under some reducible representation $R = r_1 \oplus r_2 \oplus \cdots \oplus r_k$, where r_i are distinct irreducible reps, then there are k operators, $\mathcal{O}_1, \ldots, \mathcal{O}_k$ that do not mix with each other. Moreover, $\mathcal{O}_i$ only mixes with other operators that transform as rep r_i under G.

Example 1:

Consider again our operator $\mathcal{O}_1 = (\bar{s}\gamma^\mu P_L c)(\bar{u}\gamma_\mu P_L d) \equiv (\bar{s}c)(\bar{u}d)$ for short.

The QCD lagrangian includes

$$\mathcal{L} = \bar{c}_L i\not{D} c_L + \bar{c}_R i\not{D} c_R + \bar{d}_L i\not{D} d_L + \bar{d}_R i\not{D} d_R - m_c(\bar{c}_L c_R + \bar{c}_R c_L) - m_d(\bar{d}_L d_R + \bar{d}_R d_L)$$

In the limit $m_c \to 0$, $m_d \to 0$ this exhibits an $SU(2)_R \otimes SU(2)_L$ symmetry

$$\begin{pmatrix} d_L \\ c_L \end{pmatrix} \to V_L \begin{pmatrix} d_L \\ c_L \end{pmatrix}, \quad \text{idem } L \to R, \quad \text{with } V_L \subset SU(2)_L \quad V_R \subset SU(2)_R$$

Under $SU(2)_L$ $\mathcal{O}$ is in $2 \otimes 2 = 1 \oplus 3$ rep:

1: $\tilde{\mathcal{O}}_1 = (\bar{s}c)(\bar{u}d) - (\bar{s}d)(\bar{u}c)$

3: $\tilde{\mathcal{O}}_2 = (\bar{s}c)(\bar{u}d) + (\bar{s}d)(\bar{u}c)$

We know $\mathcal{O}_1$ and $\mathcal{O}_2 = (\bar{s}T^a c)(\bar{u}T^a d)$ are closed under renormalization. But $\mathcal{O}_1 = \frac{1}{2}(\tilde{\mathcal{O}}_1 + \tilde{\mathcal{O}}_2)$. So this is just a change of basis that diagonalizes $\gamma_{\mathcal{O}}(g)$ (to all orders in g).

[Parenthetical remark:

To see this more explicitly, put color indices back in an Fierz rearrange:

For $\tilde{O}_{1,2}$ use: $(\bar{s}d)(\bar{u}c) = (\bar{s}_i d^i)(\bar{u}_j c^j) = (\bar{s}_i c^j)(\bar{u}_j d^i)$

For O_2 use $T^a_{ij} T^a_{mn} = \frac{1}{2}(\delta_{in}\delta_{mj} - \frac{1}{3}\delta_{ij}\delta_{mn})$

$$\text{so} \quad O_2 = -\frac{1}{6}\underbrace{(\bar{s}_i c^i)(\bar{u}_j d^j)}_{O_1 = \frac{1}{2}(\tilde{O}_1 + \tilde{O}_2)} + \frac{1}{2}\underbrace{(\bar{s}_i c^j)(\bar{u}_j d^i)}_{\frac{1}{2}(-\tilde{O}_1 + \tilde{O}_2)}$$

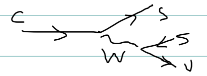
$$= \tilde{O}_1\left(-\frac{1}{12} - \frac{1}{4}\right) + \tilde{O}_2\left(-\frac{1}{12} + \frac{1}{4}\right)$$

$$\underline{O_2 = -\frac{1}{3}\tilde{O}_1 + \frac{1}{6}\tilde{O}_2}$$

End parenthetical remark]

Example 2:

Now consider $O = (\bar{s}c)(\bar{u}s)$ as you would for



Again $\dim(O) = 6$,

As before we can use $U(1)$ quantum numbers to tell us that the closed set has operators with $\bar{u}_L$ and c_L. But not $\bar{s}_L$ & s_L since this is invariant under s-number (left or right).

So we are looking for dim-6 operators with $\bar{u}_L$ & c_L. That is dim 3 combinations of fields that can make a Lorentz scalar with $\bar{u}_L$ & c_L:

Let's list dim 3 ops
- $\psi' \bar{\psi}$ where ψ' & ψ are any two fermions
- $D_\mu D_\nu D_\lambda$, including, eg $D_\lambda [D_\nu, D_\lambda] = ig D_\mu G_{\nu\lambda}$ ($+ ie D_\mu F_{\nu\lambda}$ if QED is retained).

So the change from $(\bar{s}c)(\bar{u}d)$ to $(\bar{s}c)(\bar{u}s)$ seems to have tremendously complicated the problem. Let's limit the possibilities using symmetry.

- In the absence of QED (good enough for now), if ψ' & $\bar{\psi}$ are leptons they are effectively a c-number, hence of $\dim = 0$ rather than $\dim = 3$ $\Rightarrow$ no mixing $\Rightarrow$ ignore leptons (or, diagramatically, need photons to get quarks to interact with leptons).
- Consider possibilities for $\psi' \bar{\psi} \to q \bar{q}$ (no leptons). Now $\mathcal{L}_{eff}$ for $E < M_W$ contains 5 quark flavors, u, d, s, c, b, so we have a huge symmetry at our disposal:

$$SU(5)_L \otimes SU(5)_R$$

Now O is trivial under $SU(5)_R$ and transforms as $5 \times \bar{5} \times 5 \times \bar{5}$ under $SU(5)_L$

$$5 \times \bar{5} = 1 + \underset{\uparrow \text{Adjoint}}{24} \qquad \Rightarrow (1+24) \otimes (1+24) = \overset{\text{no singlets since } c\,\bar{u}}{\cancel{1}} + 24 + 24 + 24 \otimes 24$$

Total overkill (would have to find weight in $SU(5)_L$ of $\bar{u}_L c_L \bar{s}_L s_L$ and find how many irreduc. reps in $(5,\bar{5})^2$ contain it).

Instead look at dim-3 operator multiplying $c_L \bar{u}_L$: since it transforms like $s_L \bar{s}_L$, it is $G' = SU(4)_L \times U(1)_{s_L} \times SU(5)_R$ invariant

(i) If make up of D_μ's, this is already G' invariant. Since we have to combine $\bar{u}_L$ & c_L with vectors to make a Lorentz invariant u & c come in as $\bar{u}_L \gamma^\lambda c_L$

Derivatives have to act on something so

$$\bar{u}_L \gamma_\lambda D_\mu D_\nu D_\rho c_L \; T^{\lambda\mu\nu\rho}$$

where $T^{\lambda\mu\nu\rho}$ is a 4-index (Lorentz) invariant tensor:

$T^{\lambda\mu\nu\rho}$: $\eta^{\lambda\mu}\eta^{\nu\rho}$, $\eta^{\lambda\nu}\eta^{\mu\rho}$, $\eta^{\lambda\rho}\eta^{\mu\nu}$ or $\epsilon^{\lambda\mu\nu\rho}$

Simplify:

- $\epsilon^{\mu\nu\lambda\sigma}$ is CP odd.
- Use equations of motion $\not{D}\psi = 0$ and integration by parts, so, eg

$$\bar{u}_L \not{D} D^2 c_L = \underbrace{\partial_\mu (\bar{u}_L \gamma^\mu D^2 c_L)}_{\partial_\mu \to P_\mu = 0 \text{ since } \int d^4x \, \mathcal{L}_{eff}} - \underbrace{\bar{u}_L \overleftarrow{\not{D}} D^2 c_L}_{0 \text{ by EOM}}$$

$$\Rightarrow \text{only } \bar{u}_L D^\mu \not{D} D_\mu c_L = \bar{u}_L D^\mu [\not{D}, D_\mu] c_L = \bar{u}_L D^\mu \gamma^\nu [D_\nu, D_\mu] c_L = -ig \bar{u}_L \gamma^\nu D^\mu G_{\mu\nu} c_L$$

$$= \bar{u}_L [D_\mu, D_\nu] \gamma^\nu D^\mu c_L = ig \bar{u}_L \gamma^\nu G_{\mu\nu} D^\mu c_L$$

So $\bar{u}_L D^\mu \not{D} D_\mu c_L = \frac{1}{2} \left(\bar{u}_L \gamma^\nu T^a c_L (D^\mu G_{\mu\nu})^a + (1-1) ig \bar{u}_L \gamma^\nu G_{\mu\nu} D_\mu c_L \right)$

$\Rightarrow$ The only operator is $\bar{u}_L \gamma^\nu T^a c_L (D^\mu G_{\mu\nu})^a$

By EOM, $D^\mu G^a_{\mu\nu} \propto \sum_{\text{all quarks}} \bar{\psi} \gamma_\nu T^a \psi$ $\Rightarrow$ operator $= \sum_{u,d,s,c,b} \bar{u}_L \gamma^\mu T^a c_L \, \bar{\psi} \gamma_\mu T^a \psi$

(ii) If made of a quark bilinear: $SU(5)_R$ scalar $\rightarrow$ either $\bar{\psi}_L \psi'_L$ or $\sum_i \psi_{iR} \bar{\psi}_{iR}$

- $\psi_L \bar{\psi}'_L$: $SU(4)_L \times U(1)_{s_L}$ invariant

$$\sum_{\psi \neq s} \bar{\psi}_L \gamma^\mu \psi_L \; , \; \sum_{\psi \neq s} \bar{\psi}_L \gamma^\mu T^a \psi_L \; ; \; \bar{s}_L \gamma^\mu s_L , \; \bar{s}_L \gamma^\mu T^a s_L$$

But can replace $\sum_{\psi \neq s}$ with $\sum_{\text{all } \psi}$.

- ψ_R $\sum_{\text{all } \psi} \bar{\psi}_R \gamma^\mu \psi_R$, $\sum_{\text{all } \psi} \bar{\psi}_R \gamma^\mu T^a \psi_R$

Summary :

$$O_1 = \bar{u}_L \gamma^\mu c_L \, \bar{s}_L \gamma_\mu s_L$$ (or equivalent bases, as studied earlier)

$$O_2 = \bar{u}_L \gamma^\mu T^a c_L \, \bar{s}_L \gamma_\mu T^a s_L$$

$$O_3 = \bar{u}_L \gamma^\mu c_L \left(\bar{u}_L \gamma_\mu u_L + \bar{d}_L \gamma_\mu d_L + \cdots + \bar{b}_L \gamma_\mu b_L \right)$$

$$O_4 = \bar{u}_L \gamma^\mu T^a c_L \left(\bar{u}_L \gamma_\mu T^a u_L + \bar{d}_L \gamma_\mu T^a d_L + \cdots + \bar{b}_L \gamma_\mu T^a b_L \right)$$

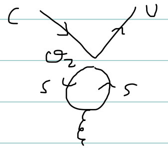
$$O_5 = \bar{u}_L \gamma^\mu c_L \left(\bar{u}_R \gamma_\mu u_R + \bar{d}_R \gamma_\mu d_R + \cdots + \bar{b}_R \gamma_\mu b_R \right)$$

$$O_6 = \bar{u}_L \gamma^\mu T^a c_L \left(\bar{u}_R \gamma_\mu T^a u_R + \bar{d}_R \gamma_\mu T^a d_R + \cdots + \bar{b}_R \gamma_\mu T^a b_R \right)$$

What diagrams produce the mixing?

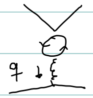
We know $O_1 \leftrightarrow O_2$ from  +  + ⋯ (6 graphs)

But now we can make a loop of s :



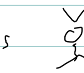
→ Produces $\bar{u}_L \gamma^\mu T^a c_L \left(D^\nu G_{\mu\nu} \right)^a$

How is this $\bar{u}_L \gamma^\mu T^a c_L \left(\bar{u} T^a \gamma_\mu u + \cdots + \bar{b} T^a \gamma_\mu b \right) = O_4 + O_6$?

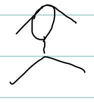


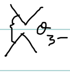
seems to be non-local, i.e. $\frac{1}{q^2}$ from propagator, but  ~ $\left(q_\mu q_\lambda - q^2 \eta_{\mu\lambda} \right)$

and  ~ $\bar{u}(p') \gamma^\mu u(p)$ so $q_\mu \bar{u}(p') \gamma^\mu u(p) = \bar{u}(p') (\not{p} - \not{p}') u(p) = 0$

and for the second term, $-q^2 \eta_{\mu\lambda}$, the $\frac{1}{q^2}$ from propagator makes  local!

Operators $O_3 - O_6$ are called "penguin operators". The reason is a peculiar story, that someone (John Ellis) made a bet he could use any term in his next paper, and the challenge was to use "penguin" (well, that's my understanding).

 a "penguin" ??

Of course  O_{3-6} +  O_{3-6} + ⋯ +  $O_3 - O_6$ contribute to the full 6x6 γ_O matrix.

Politzer, NPB 172 (1980) 349

5.4 EOM in matrix elements.

How do we justify using EOM in matrix elements? After all in the functional integral formulation of QFT, $\int [dA_\mu][d\bar\psi][d\psi]\, e^{iS}$ includes a sum overall field configurations, not just those that satisfy the EOM ?

If $S[\phi] = \int d^4x\, \mathcal{L}(\phi, \partial_\mu\phi)$ is the action integral, then we define $\mathcal{E}(\phi) = \frac{\delta S}{\delta\phi(x)}$, that is

$$S[\phi+\delta\phi] = S[\phi] + \int d^4_x \frac{\delta S}{\delta\phi(x)}\delta\phi(x) + \mathcal{O}(\delta\phi)^2$$, so $\mathcal{E}(\phi)=0$ is the EOM.

An argument due to Politzer:

$$Z(J) = \int [d\phi]\, e^{iS[\phi] + i\int d^4_x\, J\phi}$$

Change variables $\phi = \phi(\phi')$, $[d\phi] = \det\left(\frac{\delta\phi(x)}{\delta\phi'(y)}\right)[d\phi']$, $S[\phi] = S[\phi(\phi')]$

And define $\phi(\phi') = \phi' + \chi\, F(\phi')$ where $\chi = \chi(x)$ is a c-function, and $F(\phi)$ is some arbitrary polynomial function of ϕ' that may contain derivatives. Then if χ is infinitesimal (we'll take $\delta/\delta\chi(x)$ and set $\chi=0$), and dropping the prime

$$Z(J) = \int [d\phi]\, \det(1+\chi F')\, e^{iS[\phi] + i\int J(\phi+\chi F) + i\int \chi F(\phi)\mathcal{E}(\phi) + \mathcal{O}(\chi^2)}$$

Now, we can freely change $i\int J(\phi+\chi F) \to i\int J\phi$, which defines different Z but same physical amplitudes (ie, S-matrix). I should denote the new Z by a new symbol, say $\tilde{Z}(J)$, but I won't (since it is irrelevant for S-matrix).

Now the point is that

$\frac{\delta Z(J)}{\delta\chi(x)} = 0$ because it never depended on χ. If we could ignore the det factor this would mean

$$0 = \int [d\phi]\, e^{iS + \int J\phi}\, F(\phi)\mathcal{E}(\phi)$$

In other words, if $\mathcal{O} = F\mathcal{E}$ is an operator that vanishes by EOM then

$$\langle \psi_f | \mathcal{O} | \psi_{in} \rangle = 0$$

So we need to establish that the Jacobian

$$J = \left|\det \frac{\delta\phi(x)}{\delta\phi'(y)}\right|$$ does not contribute to matrix elements.

Now,
$$\frac{\delta\phi(x)}{\delta\phi'(y)} = \frac{\delta}{\delta\phi'(y)}\left(\phi'(x)+\chi(x)F(\phi'(x))\right) = \left[1+\chi(x)F'(\phi'(x))\right]\delta^4(x-y)$$

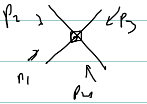
and $\left|\det\frac{\delta\phi}{\delta\phi'}\right| = \exp\left(\mathrm{Tr}\ln\frac{\delta\phi}{\delta\phi'}\right) = \exp\left(\int d^4x \ln\left[1+\chi(x)F'(\phi'(x))\right]\delta^4(0)\right)$

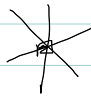
so $\displaystyle\frac{\delta}{\delta\chi(x)}\left|\det\frac{\delta\phi}{\delta\phi'}\right|\Bigg|_{\chi=0} = F'(\phi'(x))\,\delta^4(0)$

In dim reg $\delta^4(0)\to\int d^dk = 0$, so we can ignore this.
Alternatively, this goes away by normal ordering, it arises from self-contraction of $\mathcal{E}F$.

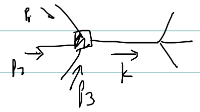
Example: $\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4$ $\qquad \mathcal{E}(\phi) = -(\partial^2+m^2)\phi - \frac{1}{6}\lambda\phi^3$

Take $\theta = F\mathcal{E}$ with $F=\phi^3$

So, eg 

$= i\,3!\sum_{i}^{4}(p_i^2-m^2)$  $= -i\frac{6!}{6}\lambda$

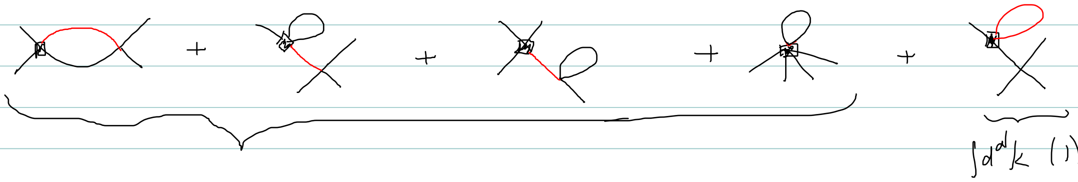
Tree level 6 particle amplitude (amputated + on-shell external legs)



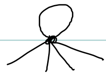
$i(-i\lambda)\frac{4\cdot 6!}{4!}(k^2-m^2)\frac{i}{k^2-m^2} = i\,5!\,\lambda$ which cancels 

(The combinatorics is $\phi^3(-\partial^2-m^2)\overline{\phi \times \frac{1}{4!}\phi}{}^4 \to \frac{4}{4!}\phi^6$ to be contracted with $\phi_1\cdots\phi_6 \to \frac{4}{4!}6!$).

At 1-loop, 4-pt amplitude (2→2 scattering) (red $=(p^2-m^2)\frac{i}{p^2-m^2}=1$)



$\int d^dk\,(1) = \delta^{(d)}(0)\to 0$


(1st three cancel 4th)

(or $:\phi^3(\partial^2+m^2)\phi:$ has no self contraction $:\overbrace{\phi\phi}\,\phi(\partial^2+m^2)\phi: = 0$).

Exercise: It is necesary that $F(\phi)$ be at least quadratic in fields. Why? what goes wrong if F is linear in ϕ?

5.5 Loose ends

5.5.1 NLL.

Sketch: $C(M) = \left(1 + d_1 \frac{\bar\alpha(M)}{\pi}\right) c_0$

$$C(\mu) = e^{\int_{\bar g(M)}^{\bar g(\mu)} \frac{dg'}{g'}\left(2a_0 + 2a_1 \frac{g'^2}{16\pi^2}\right)} C(M) = e^{a_0 \ln\frac{\bar\alpha(\mu)}{\bar\alpha(M)} + a_1\left(\frac{\bar\alpha(\mu)}{4\pi} - \frac{\bar\alpha(M)}{4\pi}\right)} C(M)$$

$$C(\mu) = \left(\frac{\bar\alpha(\mu)}{\bar\alpha(M)}\right)^{a_0}\left(1 + a_1\left(\frac{\bar\alpha(\mu)}{4\pi} - \frac{\bar\alpha(M)}{4\pi}\right)\right)\left(1 + d_1 \frac{\bar\alpha(M)}{\pi}\right) c_0$$

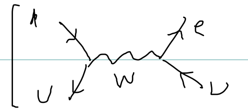
Note 1-loop running ⟺ tree level matching

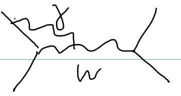
2-loop running ⟺ 1-loop matching

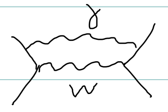
etc.

5.5.2 In general, "matching" is the difference between full & EF theories at $\mu = M$ (or $\mu \sim M$, not necesarily =).

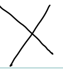
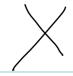
So, eg, in QED

$\Big[$ 

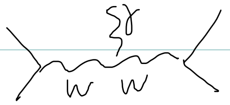
$+$ 

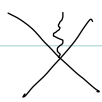
$+$ 

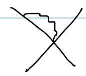
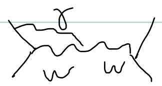
$+ \cdots \Big]_{\text{ren}}$

$- \Big[$  $+$  $+ \cdots \Big]_{\text{ren}} = d_1 \frac{\alpha}{4\pi}$ 

Note that

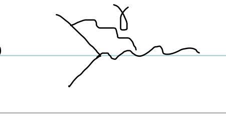


$=$  $\sim \frac{1}{M_W^4} \bar u \gamma^\mu d \bar e \gamma^\nu \nu F_{\mu\nu}$ is dim 8

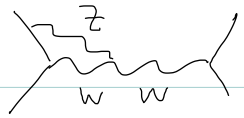
So  does <u>not</u> match onto 

. However 

is IR safe in the sense that it is finite in IR even at $p_{ext} = 0 = m_{light}$

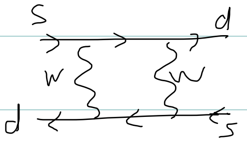
So 

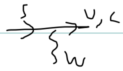
$\sim \int d^dk \underset{\uparrow\gamma}{\frac{1}{k^2}} \underset{\uparrow\psi}{\frac{1}{k}} \underset{\uparrow}{k} \frac{1}{k^2 - M_W^2} \sim \frac{1}{\epsilon} + \ln\frac{M_W}{\mu} + \text{"}1\text{"} \longrightarrow M_W$ contributes to d_1

In fact 

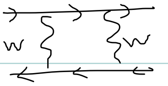
contributes to d_1 as well.

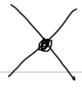
5.53 A similar situation arises in cases where the quantum numbers demand two heavy particles exchanged.

eg 

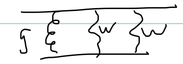
is $\Delta S = 2$ (or -2) but 

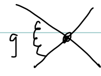
is only $\Delta S = 1$

So in principle, 

$\longrightarrow$  with ~ $\frac{g_2^4}{16\pi^2}\frac{1}{M_W^2}$ coefficient.

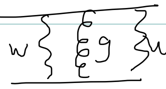
In this case QCD

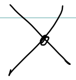


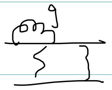
match 

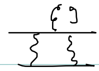
and have interesting dynamics (p-dependence)

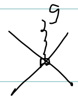
except for a short distance ~$\bar{\alpha}(M_W)$ mismatch, as before.

But 

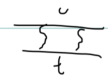
matches $\frac{\alpha_s}{4\pi}$  directly (all short distance)

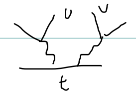
Likewise 

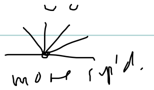
(

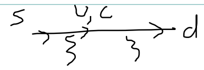
matches the dim 8 op 

)

(likewise for 

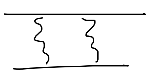
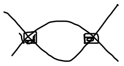
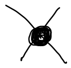
$\rightarrow$ 

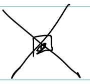
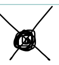
$\rightarrow$ 
more sup'd. ...)

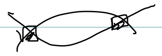
Monkey wrench! In SM (ignore t) 

~ $\sum_{i=u,c} V_{is}V_{id}^* \gamma^\mu P_L \frac{1}{\not{k}-m_i}\gamma^\nu P_L$

But $VV^\dagger = 1$, ie, $\sum_{i=u,c} V_{is}V_{id}^* = 0$, so at $m_i = 0$ the diagram vanishes. So propagator is ~ $_iV_{cs}V_{cd}^*(m_c - m_u)\frac{1}{k^2}$. So in  we take $\frac{\partial}{\partial p}$ as before and find

$\lim_{M_W \to \infty} M_W^{4}$ [ −  − ] $= 0$

where  $= \frac{1}{M^2} (\bar{d} c \bar{c} s + \ldots) = \mathcal{L}_{eff}^{\Delta S=1}$ and  $= \frac{1}{M^4} \bar{d} s \bar{d} s = \mathcal{L}_{eff}^{\Delta S=2}$

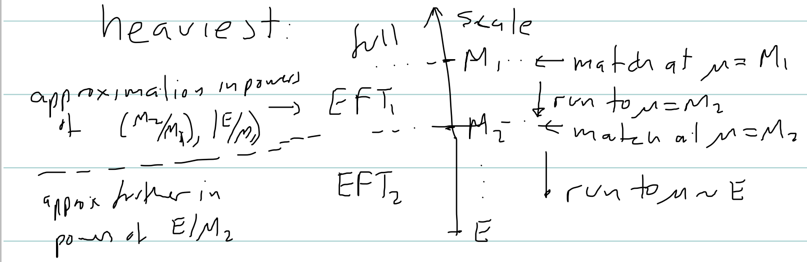
and  is just from $T(\mathcal{L}_{eff}^{\Delta S=1}(x)\, \mathcal{L}_{eff}^{\Delta S=1}(0))$

5.5.4 Multiscale problems

If instead of 2 scales, $M \gg E$, we have many widely separated heavy particle masses and a low energy physical process,

$$M_1 \gg M_2 \gg \cdots \gg E$$

we can avoid large logs, $\ln E/M_i$, in the matrix element AND large powers $(M_i/M_j)^n \gg 1$ (ie, $i<j$ and $n>0$) by constructing a sequence of EFTs by sequentially integrating out the heavy particles, starting from the heaviest:



↖ larger scale always in denominator.

This typically gives (at LL) $C(\mu) = \left(\frac{\bar{\alpha}_1(M_1)}{\bar{\alpha}_1(M_2)}\right)^{a_1} \left(\frac{\bar{\alpha}_2(M_2)}{\bar{\alpha}_2(\mu)}\right)^{a_2} C(\mu)$

where $\bar{\alpha}_i$ the running coupling of EFT_i

Exercise: Find γ_0 at 1-loop ! for the operator that occurs in $K^0-\bar{K}^0$ mixing, $(\bar{s}_L\gamma^\mu d_L)(\bar{s}_L\gamma^\mu d_L)$. Estimate the corrections from this LO short distance QCD effect in the formula for ϵ_K.