Mini Course on

Effective Field Theory (EFT)

IFT / UAM

2022 — November 8 - December 2

Introduction

There are many approaches to EFT. They all rely on the observation that in quantum field theory very heavy exitations of fields can more or less be ignored when looking at processes which involve low energies.

By "very heavy" exitations we mean those that carry high energy, higher than the energy of the process(es) under investigation.

But what does this mean? After all, energy is frame dependent (not a Lorentz invariant quantity).

In the first instance we will look at exitations that are heavy because they carry a large mass, M. Then (energy of excitation) $\geq M$, and we can demand that the processes we are interested in have (total energy) $<< M$ in some frame, and therefore in many.

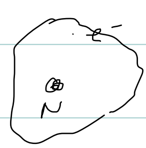
In other instances the very definition of the EFT will be frame dependent and the separation of heavy – light modes arbitrary (in the sense that they shift around as one changes frames). As we will see, this is still useful.

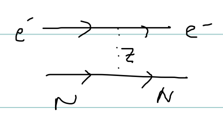
Rather than attempting a very generic description of EFTs I prefer quickly plunging into specifics — I think this helps understand the meaning and usefulness better.

So let's look at some issues that come up in QFT and how EFT help to address them.

Above I said that heavy exitations can be "more or less" ignored at low energies. Let's look at this in the prototypical case: the weak interactions. Recall these have mediators, the $W^\pm$ & Z^0 vector bosons, that have masses ~ 100 GeV (I use $c=1$ and $\hbar=1$ in these lectures, so mass $= 100$ GeV means 100 GeV$/c^2$).

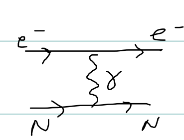
If we are concerned with atomic physics, or with e^-e^+ scattering at, say, $E_{CM} \lesssim 1$ GeV, then we can safely ignore the weak interactions.





$$\sim \frac{1}{q^2 - M_Z^2}$$

vs



$$\sim \frac{1}{q^2}$$

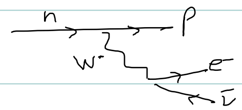
$$\text{ratio} \left|\frac{\text{weak}}{\text{e.m.}}\right| \sim \left|\frac{q^2}{q^2 - M_Z^2}\right| \sim \frac{(\text{eV})^2}{(100\text{ GeV})^2} \sim 10^{-22}$$

This is the "more" in "more or less"

Now for the "Less".

Nuclei <u>do</u> β-decay. So does the neutron. And $\mu^\mp$.

In



the $q^2 = (p_e + p_\nu)^2 = (p_n - p_p)^2$ in $\dfrac{1}{q^2 - M_W^2}$ can be neglected

but we do not ignore the whole process!

What we may do is $\dfrac{1}{q^2 - M_W^2} \longrightarrow -\dfrac{1}{M_W^2}$

describe the interaction as a "contact term", that is, a local interaction.

In equations

$$\text{transition amplitude} = i\mathcal{M}(n \to p e^- \bar\nu) = i^3 g^2 \langle p e^- \bar\nu | \int d^4x \, T\big(J^\mu_{had}(x)\, \Delta_W(x-0)_{\mu\nu}\, J^\nu_{lep}(0)\big) | n \rangle$$

$$\simeq -i g^2 \frac{1}{M_W^2} \langle p e^- \bar\nu | J^\mu_{had}(0)\, J_{lep\,\mu}(0) \; | n \rangle$$

as if $\mathcal{L}_{int}(x) = -g^2 \dfrac{1}{M_W^2} J^\mu_{had}(x) J_{lep\,\mu}(x)$, "local"

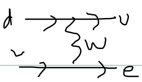
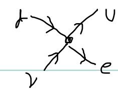
(Here $J^\mu_{had}(x) = \bar p(x)\,(g_V \gamma^\mu + g_A \gamma^\mu \gamma_5)\, n(x)$ $\qquad J^\mu_{lep}(x) = \bar e(x)\, \gamma^\mu (1-\gamma_5)\, \nu(x)$

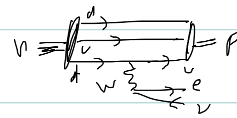
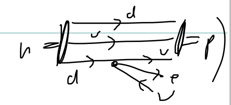
$\uparrow$ don't get distracted by this, focus on local vs non-local)

This, of course, is the famous Fermi theory: "4 fermion interaction"

So while we cannot ignore the heavy field altogether, we can describe its effect by introducing a local interaction.

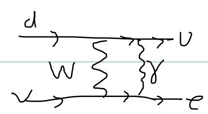
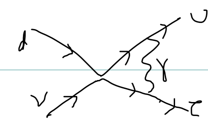
Incidentally, we may as well write $J^\mu_{had}(x) = \bar u(x) \gamma^\mu (1-\gamma_5)\, d(x)$ and it is still true that $\mathcal{M}(n \to p e^- \bar\nu) = -g^2 \frac{1}{M_W^2} \langle p e^- \bar\nu | J^\mu_{had}(0) J_{lep\,\mu}(0) | n \rangle$

so that we picture  $\approx -g^2 \frac{1}{M_W^2}$ 

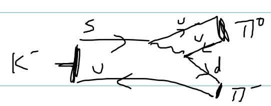
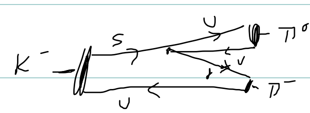
(and in the matrix element between baryons:  $\sim$ )

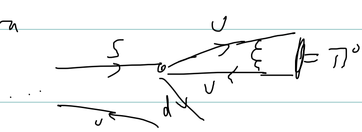
Some of the issues that arise:

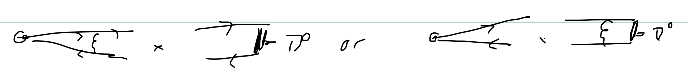
- Radiative corrections

  ★  is finite, but  is ∞. How do we handle this? The 4-fermion term is not renormalizable. We had a predictive theory (renormalizable = finite # of parameters), but replaced it by a non-renormalizable one, non-predictive (∞ # of parameters).

  ★ Consider a W-exchange between J^μ_{had} and itself (or, rather, $J^{\mu\dagger}_{had}$). This could be relevant for, say $K^- \to \pi\pi^0$ decay, as in

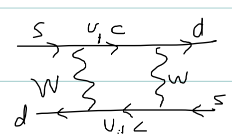
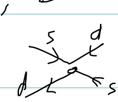
 $\longrightarrow$ 

Ahora  Is this gluon exchange part of a correction to $\mathcal{L}_{int}$, or part of π^0, ie

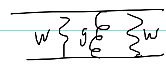
 ?

★ Scale: If we compute these radiative corrections, do we use $\alpha(M_W)$ or $\alpha(M_n)$ (or $\alpha_s(M_W)$ vs $\alpha_s(M_K)$ in the 2nd example)? "Scale uncertainty".

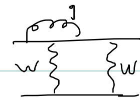
★ What about processes that require exchange of 2 heavy quanta (example, the case in K^0-$\bar{K}^0$ mixing, $\Delta S = 2$):

 $\stackrel{??}{=}$ 

* If this is correct, how do we include



or



??

There does not seem to be a corresponding graph on the local interaction version

We'll address these problems today. We will get into the guts of how it all works:

The scale uncertainty problem derives from having disparate scales. The technique we'll utilize to approach this is the effective field theory (EFT). It allows one to look at the physics of the shortest distance/time scales ignoring the longer ones, and then moving sequentially to longer distance/time.

The problems we are facing are artifacts of perturbation theory.
For example, if we could compute non-perturbatively (or at least perturbatively to all orders) we would use $\alpha_s(\mu)$ for the coupling (together with other $y(\mu)$, say). And the (physical) amplitudes would actually be μ-independent. Of course this is the content of the renormalization group equation (RGE), which we'll use extensively.

There is a related problem worth investigating. Disparate scales often result in possible breakdown of perturbation theory. The best example is in grand unified theories (GUTs) for which M_{GUT} can be $10^{15}\, v$, $v = v_{SM} = 250\,\text{GeV}$

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Review:

To set the stage, consider $SU(5)$ grand-unification.

This is a Yang-Mills theory with gauge group $SU(5)$ that breaks spontaneously to $SU(3)\times SU(2)\times U(1)$

Gauge fields are in adjoint representation. If

Ψ_i $i=1,\dots,5$ is a vector in the fundamental (defining) rep

$\Psi \to U\Psi$ with $U^\dagger U = 1$, U a 5×5 matrix, $U = e^{i\omega^a T^a}$ (ω^a real parameters, T^a generators)

$U^\dagger U = 1 \Rightarrow T^{a\dagger} = T^a$, $\det U = 1 \Rightarrow \operatorname{Tr} T^a = 0$. $a = 1,\dots,N^2-1$ for $SU(N)$.

$$T^a = \begin{pmatrix} \frac{1}{2}\lambda^a & 0 \\ 0 & 0 \end{pmatrix} \quad \lambda^a\ a=1,\dots,8 \ \{SU(3)\ ; \qquad T^a = \begin{pmatrix} 0 & 0 \\ 0 & \frac{\sigma^a}{2} \end{pmatrix} \quad \sigma^{1,2,3}\ \text{Pauli matrices}\ \{SU(2)$$

$$T^{24} = \frac{1}{2\sqrt{15}}\operatorname{diag}(2,2,2,-3,-3) \text{ gives } U(1) \quad ; \text{ normalized to } \operatorname{Tr} T^a T^b = \tfrac{1}{2}\delta^{ab}$$

Write $\Psi = \begin{pmatrix} d^c \\ \ell \end{pmatrix}$ (3, 2) and consider $D_\mu \Psi = \partial_\mu \Psi + i g_5 A^a_\mu T^a \Psi$

Contains
$$i g_3 A^a_\mu \tfrac{1}{2}\lambda^a d^c + i g_2 W^a_\mu \tfrac{1}{2}\sigma^a \ell + \underbrace{i\hat{g}_1 B_\mu \frac{1}{2\sqrt{15}}\left(2 d^c - 3\ell\right)}_{i g_1 B_\mu \left(\frac{1}{3} d^c - \frac{1}{2}\ell\right)} \qquad \left(g_1 = \frac{3}{\sqrt{15}}\hat{g}_1\right)$$

where, really, $g_3 = g_2 = \hat{g}_1 = g_5 = g_{GUT}$

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If you compute, say, $e^+e^-\to\mu^+\mu^-$ in the GUT in terms of its coupling constant, g_{GUT}, you'll find to 1-loop that

$$a = a_{born}\left(1 + c\frac{\alpha_{GUT}}{\pi}\ln\frac{M_{GUT}^2}{v^2}+\dots\right)$$

Here c = some number $\mathcal{O}(1)$, and I've omitted stuff that does not contain $\ln\frac{M_{GUT}^2}{v^2}\sim 70$

Now $\alpha_{GUT}\sim\frac{1}{40}$ (is fairly typical) and c can easily be more than π (if not for this process, for some of the great many low energy processes in the PDG book).

Not only is the 1-loop correction large, $\sim\mathcal{O}(100\%)$, at n-loops there will be a correction of order $\left(\frac{\alpha_{GUT}}{\pi}\ln\frac{M_{GUT}^2}{v^2}\right)^n$.

If you can account for all the terms of the form $\left(\frac{\alpha_{GUT}}{\pi}\ln\frac{M_{GUT}^2}{v^2}\right)^n$, say by summing the corresponding $\sum_n C_n\left(\frac{\alpha_{GUT}}{\pi}\ln\frac{M_{GUT}^2}{v^2}\right)^n$, then the next order gives corrections of the form $\sum_n C_n'\frac{\alpha_{GUT}}{\pi}\left(\frac{\alpha_{GUT}}{\pi}\ln\frac{M_{GUT}^2}{v^2}\right)^n$. If $\frac{\alpha_{GUT}}{\pi}\ln\frac{M_{GUT}^2}{v^2}\sim 1$ then these subleading corrections are of order $\frac{\alpha_{GUT}}{\pi}\sim\frac{1}{\ln\frac{M_{GUT}^2}{v^2}}\sim\frac{1}{70}$. Nice. All we need to do to get per-cent accuracy is to sum these "leading-logs."
But failing to do so we incur in 100% errors.

The EFT technique takes advantage of the simpler form of the RGE when there is only one relevant scale (one at a time!) in the problem to sum the leading-logs (LL) and if needed the next-to-LL (NLL) ie $\alpha\left(\alpha\ln\frac{M^2}{v^2}\right)^n$, etc.

Note: There is no WIKI page for this, Here is your chance to make a mark!

2. Appelquist - Carrazzone Decoupling Theorem.

Not a mathematician ...

Consider a theory with $\mathcal{L}_{full} = \mathcal{L}_{light} + \frac{1}{2}\left[(\partial_\mu\phi)^2 - M^2\phi^2\right] + \mathcal{L}_{\phi\text{-light}}$

$\mathcal{L}_{light}$ may involve many fields, but all of mass $<< M$. It depends on parameters g_i (and m_i). $\mathcal{L}_{\phi\text{-light}}$ has the interactions between ϕ and light-field and depends on g_i and possibly additionally on coupling h_i.

Consider Green functions $G^{(n)}(p_1, ..., p_n)$ (or better yet, amplitudes) of n light particles (associated with the light fields), restricted to $|p_i| << M$. Then

$$G^{(n)}(p_1, ..., p_n) = Z^{n/2}\, \tilde{G}^{(n)}(p_1, ..., p_n)\left(1 + O\left(\frac{1}{M}\right)\right)$$

where $\tilde{G}^{(n)}$ is computed from

unrenormalizable lagrangian $\qquad \mathcal{L}_{eff} = \tilde{\mathcal{L}}_{light}$

where $\tilde{\mathcal{L}}_{light}$ is constructed out of the fields in $\mathcal{L}_{light}$, with new (possibly more) couplings $\tilde{g}_i$. The $\tilde{g}_i = \tilde{g}_i(g_i, M)$ and $Z = Z(g_i, M)$, are not functions of momenta and are universal (the same choice of $\tilde{g}$ and Z for any Green function).

The meaning is clear, heavy particles appear in $G^{(n)}$ only through virtual effects, by construction. At large M ($M >> m, p$) the effects of M decrease as $\left(\frac{1}{M}\right)^{\#}$, except when M appears in logs. The content of the decoupling theorem is that (i) there are no positive powers of M, and (ii) The $\log M$ terms can all be absorbed into $\tilde{g}$ and Z.

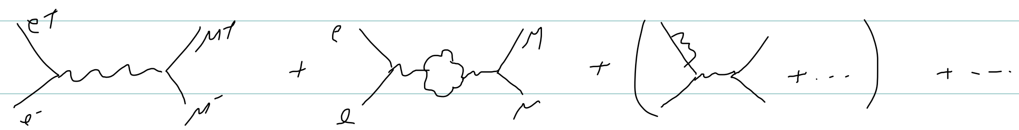
For the theorem to work you have to be able to take M arbitrarily large holding g_i constant. It fails when $M = gv$ because either $v \to \infty$ and all particles get heavy, or $g_i \to \infty$ together with M, so the $O(\frac{1}{M})$ corrections can go as $O(\frac{g}{M}) = O(\frac{1}{v}) = $ fixed.

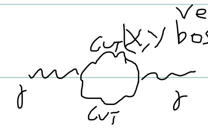
Added note on decoupling:

$v \to \infty$ is fine even if $m_{light} = y v$ (yukawa) or $m^2_{light} = \lambda v^2$ (scalar quartic) provided $y \to 0$ and/or $\lambda \to 0$ keeping m_{light} fixed.

This is what we do for, say, electroweak interactions.

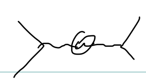
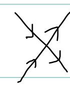
Think GUTs, say, SU(5). We can apply decoupling to the M_{GUT} heavy fields. Then by construction $\mathcal{L}_{eff}$ is just the SM with couplings $\tilde{g}_{1,2,3} = \tilde{g}_{1,2,3}(g_{GUT}, \lambda_{GUT})$. What's going on is that in



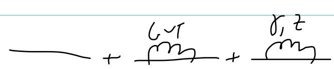
the effect of 

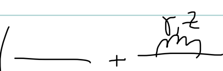
differentiates the photon (or Z) from gluons by differences in $\ln\frac{M_{GUT}^2}{\mu^2}$. The γ (or Z) (or g) self-energy $q^2\Pi(q)$ has

$\Pi(q^2) \sim \frac{\alpha}{\pi}\left[c_1 \ln\frac{M_{GUT}^2}{\mu^2} + c_2 \ln\frac{q^2}{\mu^2} + c_3 + \mathcal{O}\left(\frac{q^2}{M_{GUT}^2}\right)\right]$ where $\ln\frac{M_{GUT}^2}{\mu^2} \to$ from heavy loops, $\ln\frac{q^2}{\mu^2} \to$ from light loops. So neglecting the $\frac{q^2}{M^2}$, the new coupling has been shifted by $\ln\frac{M_{GUT}}{\mu^2}$:

 $= \times \cdot \dfrac{g_{GUT}^2}{q^2(1+\Pi)} \sim \underbrace{\dfrac{g_{GUT}^2}{\left(1+\frac{\alpha}{\pi}\ln\frac{M_{GUT}}{\mu}\right)}}_{=e^2} \cdot \underbrace{\dfrac{1}{q^2(1+\bar{\Pi})}}_{\text{obtained from } \mathcal{L}_{eff}} \cdot$ 

The external e or μ has some self energy correction that can also be broken into contributions from the heavy, that go into $Z^{1/2}$, and contributions from the Light, that are produced by $\mathcal{L}_{eff}$.



$= \left(1 + \frac{\alpha_{GUT}}{\pi}\ln\frac{M}{\mu}\right)\Big($  $\Big) + \dots$

In a way, this is just a factorization theorem (but not really, since $\tilde{g}$ is every-where).

2.1 RGE (Renormalization Group Equation) and "running" & "matching"

So the above diagramatic disussion explains how different coupling constant arise in the low energy EFT for a GUT. But there is something unsatisfactory in that presentation: it requires that we compute loops with heavy particles to get, say, $\sigma(e^+e^- \to \mu^+\mu^-)$ at low energy.

But we know this is not right. We can compute σ at $\sim E_{cm} \sim 1\,\text{GeV}$ in QED ignoring the effects of W/Z bosons let alone X/Y vector bosons. And in fact the decoupling theorem says precisely that; for this case:

> compute in QED with coupling $\alpha_{em}\left(=\frac{e^2}{4\pi}\right)$ which implicitly is given as a function of g_{GUT} and $\ln M_{GUT}$ (and/or $g_{1,2}$ of EW theory and $\ln M_{W,Z}$).

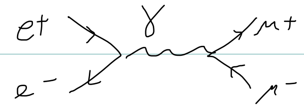
While we can then blissfully ignore that α_{em} is a function of g & $\ln M$, sometimes we would like to know what this functional dependence is. For example, for the EFT (the SM) of a GUT should have (SM = SU(3) x SU(2) x U(1) has couplings g_3, g_2, g_1)

$$g_i = g_i(g_{GUT}, M_{GUT}) \quad i=1,2,3 \qquad (\text{sorry } g_{GUT} = g_5\text{, I go back \& forth in notation})$$

3 functions of 2-parameter $\Rightarrow$ 1 relation.

So figuring out this functional dependence is interesting.

To figure this out, let's think of how coupling constants enter measurable quantities (aka, "observables"). For α_{em} we already talked about $\sigma(e^+e^- \to \mu^+\mu^-)$. We could look at $\sigma(u\bar{u} \to d\bar{d})$ for α_s, etc. Now



$\sim \frac{e^2}{s}$ $\quad s = (p_e + p_{\bar{e}})^2 = 4E_{cm}^2$ (Mandelstam variable)

At large s, $s \gg m_\mu^2$ we can ignore masses of e, μ.

So here is the plan: figure out the s-dependence of $\sigma(s)$ and use that to infer the $\ln M_{CUT}^2/\mu^2$ which we know goes into implicit dependence of α_{ew}, using knowledge about μ dependence and $\ln \mu^2/s$ (which will be explicit).

To this end use RGE as follows. By dimensional analysis

$\sigma(s, \mu, g) = \frac{1}{s} f(\mu/\sqrt{s}, g)$ (here g is any of the dimensionless coupling constants. We can also do more than one at a time)

The RGE says that in the observable quantity σ we can change the renormalization point $\mu \to \mu + \delta\mu$ and compensate with a change in $g \to g + \delta g = g + \beta(g)\frac{\delta\mu}{\mu}$ for some function $\beta(g)$ so that the physical quantities, like σ, do not change:

$$\sigma(s, \mu+\delta\mu, g+\beta\tfrac{\delta\mu}{\mu}) = \sigma(s,\mu,g) \Rightarrow \left(\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial g}\right)\sigma = 0$$

or, with $\sigma = \frac{1}{s} f(\frac{\mu}{\sqrt{s}}, g)$, $\quad \left(\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial g}\right) f = 0$

Solving the RGE

Let $dt = \frac{d\mu}{\mu}$ (saves ink), $\left(\frac{\partial}{\partial t} + \beta(g)\frac{\partial}{\partial g}\right) f(t,g) = 0$

Introduce "RGE flow": Let $\bar{g}(t,g)$ be a solution to

$$\frac{d\bar{g}}{dt} = \beta(\bar{g}) \quad \text{with boundary condition } \bar{g}(0,g) = g$$

Then $f(t,g) = f(0, \bar{g}(-t,g))$ is the solution.

To show this first notice that

$$\frac{\partial \bar{g}}{\partial g} = \frac{\beta(\bar{g})}{\beta(g)}$$. This follows from

$$dt = \frac{d\bar{g}}{\beta(\bar{g})} \Rightarrow t = \int_g^{\bar{g}(t,g)} \frac{dg'}{\beta(g')} = \int_{g+\delta g}^{\bar{g}(t,g) + \delta g \frac{\partial \bar{g}}{\partial g}} \frac{dg'}{\beta(g')}$$

$$= -\int_g^{g+\delta g} \frac{dg'}{\beta(g')} + t + \int_{\bar{g}}^{\bar{g} + \delta g \frac{\partial \bar{g}}{\partial g}} \frac{dg'}{\beta(g')}$$

So we propose that $f(t,g)$ depends on its arguments only through the combination $\bar{g}(-t,g)$: $f(t,g) = F(\bar{g}(-t,g))$. Check

$$\left(\frac{\partial}{\partial t} + \beta(g)\frac{\partial}{\partial g}\right) F(\bar{g}(-t,g)) = \frac{dF}{d\bar{g}}\frac{\partial \bar{g}(-t,g)}{\partial t} + \beta(g)\frac{dF}{dg}\frac{\partial \bar{g}(-t,g)}{\partial g}$$

$$= \frac{dF}{d\bar{g}}\left[-\beta(\bar{g}) + \beta(g)\frac{\beta(\bar{g})}{\beta(g)}\right] = 0$$

Finally, since $f(t,g) = F(\bar{g}(-t,g))$, then evaluating at $t=0$ $f(0,g) = F(\bar{g}(0,g)) = F(g)$. So the functional dependence of $F(x)$ is given by $f(0,x) \Rightarrow f(t,g) = F(\bar{g}(-t,g)) = f(0, \bar{g}(-t,g))$

Note that the functional form is not fixed by the RGE, ie, $f(0,x)$ is arbitrary.

Compare, say, with $\left(v\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) f(x,t) = 0$ having $f(x-vt, 0)$ as solutions.

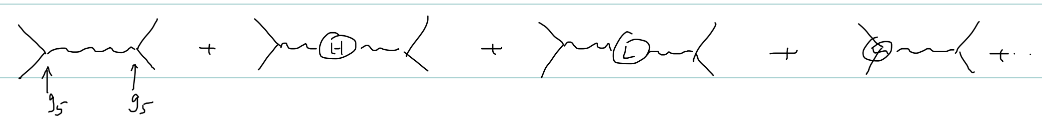
We can now use this solution in $\sigma = \sigma(e\bar{e} \to \mu\bar{\mu})$

$$\sigma = \frac{1}{s} f(\mu/\sqrt{s}, g) = \frac{1}{s} f(1, \bar{g}(\ln\tfrac{\sqrt{s}}{\mu}, g))$$

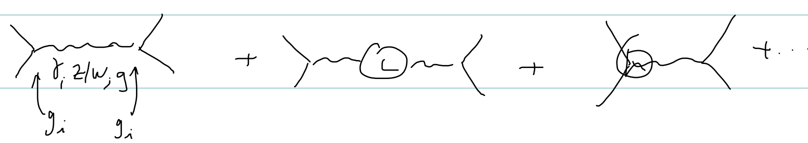
By the way the amplitude $\mathcal{M}$ ($f \propto |\mathcal{M}|^2$) also satisfies this RGE so

$$\mathcal{M}(\mu/\sqrt{s}, g) = \mathcal{M}(1, \bar{g}(\ln\tfrac{\sqrt{s}}{\mu}, g))$$

How do we use this to determine g_i in terms of g_5, M_{GUT}?
Well, we should get the same $\mathcal{M}$, up to corrections of order $E/M_{X,Y}$. But the problem is to compute reliably. So to avoid large logs take $s \sim M_{GUT}^2 \sim \mu^2$.

GUT calc $\mathcal{M}$ = 

$$= g_5^2 + g_5^4 \frac{1}{16\pi^2}(\text{constant} + \ln 1) + \cdots$$

SM calc $\mathcal{M}$ = 

$$= g_i^2 + g_i^4 \frac{1}{16\pi^2}(\text{constant}' + \ln 1) + \cdots$$

So to lowest order these agree if $\boxed{g_i = g_5}$, and recall this provided $s = \mu^2 = M_{GUT}^2$ (or approximately this, all we need is that $\frac{\alpha}{\pi}\ln\frac{M_{GUT}^2}{\mu^2} \ll 1$ and $\frac{\alpha}{\pi}\ln\frac{M_{GUT}^2}{s} \ll 1$ (and $\frac{\alpha}{\pi}\ln\frac{s}{\mu^2} \ll 1$))

Setting $g_i = g_5$ at $\mu = M_{GUT}$ means, in SM $\mathcal{M} = \mathcal{M}(1, \bar{g}_i(\ln\frac{\sqrt{s}}{M_{GUT}}, g_5))$

But since the starting point is arbitrary

$$\bar{g}_i\left(\ln\frac{\sqrt{s}}{\mu}, g_{i,\mu}\right) = \bar{g}_i\left(\ln\frac{\sqrt{s}}{M_{GUT}}, g_5\right)$$

and setting $\sqrt{s}=\mu$ $\quad \boxed{g_{i,\mu} = \bar{g}_i\left(\ln \mu/M_{GUT}, g_5\right)}$

So this is a long, round-about way of obtaining a result you probably already knew, that the coupling constants of the SM in a GUT are given by their own "running" functions $g_{i,\mu}$ with the condition that at $\mu = M_{GUT}$ they are all equal (and equal to g_{GUT}).

But this careful reasoning will be ported to many other computations AND shows what you need to do to compute corrections. More about this shortly.

Terminology:

That we fix g_{μ} at $\mu = M_{GUT}$ to equal g_5 is called "matching"

That we then compute $g_{i,\mu}$ for other μ using the RGE is called "running"

$\Rightarrow$ match & run

Let's investigate this a bit more.

Solve for $\bar{g}_i(t)$ using lowest order approximation to $\beta_i(g)$

$$\beta_i(\vec{g}) = -\frac{b_0^i}{16\pi^2} g_i^3 + \mathcal{O}(g_i^3 g_j^2) \quad \text{no sum on } i$$

Here, $b_0 = \frac{11}{3} C_2(G) - \frac{4}{3} n_f T(R_f) - \frac{1}{3} n_s T(R_s)$

where $\mathrm{Tr}\, T^a T^b = T(R)\, \delta^{ab}$ for the R-rep of group G

and for $R = \text{Adj}$ $\quad T^a T^a = C_2(G)\, \mathbb{1}$

[In general $T^a T^a = C_2(R)\, \mathbb{1}$; now $\mathrm{Tr}\, T^a T^a = C_2(R) \dim R$

and also $= T(R) \dim(\text{Adj}) \Rightarrow C_2(\text{Adj}) = T(\text{Adj}) = N$ for $SU(N)$].

Also $n_f = \#$ Dirac fermions in rep R_f, $n_s = \#$ complex scalars in R_s

(put a $\frac{1}{2}$ for Weyl or Majorana fermions, and for real scalars)

Example:

For $SU(3)$ of SM $\quad b_0^{(3)} = \frac{11}{3} \cdot 3 - \frac{4}{3} \cdot \underset{u,d,s,c,b,t}{6} \cdot \frac{1}{2} = 7$

$\uparrow T(\text{fundamental}) = 3$

Exercise: $b_0^{(2)} = \frac{19}{6}$

For $U(1)$, $C_2(G) = 0$ (Exercise: why?) and $T(R) = Q_R^2$

where Q_R is the charge under $U(1)$ transformation

for example, for $U(1)_{SM}$ (hypercharge, $Q_R \to Y$), $Y = -\frac{1}{2}$ for l_L.

Exercise: $b_0^{(1)} = -6$

Solve $\frac{d\bar{g}}{dt} = -\frac{G}{16\pi^2}\bar{g}^3 \Rightarrow \frac{d}{dt}\left(\frac{1}{\bar{g}^2}\right) = -2\frac{1}{\bar{g}^3}\left(-\frac{G\bar{g}^3}{16\pi^2}\right) = \frac{G}{8\pi^2}$

$\Rightarrow \frac{1}{\bar{g}^2(t)} - \frac{1}{g^2} = \frac{G}{8\pi^2}t$ or with $\bar{\alpha} = \frac{\bar{g}^2}{4\pi}$

$$\boxed{\frac{1}{\bar{\alpha}(t)} - \frac{1}{\alpha} = \frac{G}{2\pi}t}$$

We can use this several ways: recall $g_{i,\mu} = \bar{g}_i(\ln\frac{\mu}{M_{CUT}}, g_5)$

☆ $\alpha_i(\mu) = \bar{\alpha}_i(\ln\frac{\mu}{M_{CUT}}, g_5) = \left(\frac{1}{\alpha_5} + \frac{b_0^i}{2\pi}\ln\frac{\mu}{M_{CUT}}\right)^{-1} = \frac{\alpha_5}{1 + \frac{b_0^i\alpha_5}{2\pi}\ln\frac{\mu}{M_{CUT}}}$

which shows explicitly the dependence of g_i on α_5 & $\ln M_{CUT}$

☆ Taking the difference at t' & t we eliminate α_5:

$$\frac{1}{\bar{\alpha}(t)} - \frac{1}{\bar{\alpha}(t')} = \frac{G}{2\pi}(t-t') = \frac{G}{2\pi}\ln\mu/\mu'$$

This can be used to input $\bar{\alpha}_i(M_Z)$ and compute $\bar{\alpha}(\mu)$ as a function of μ and see at what point all 3 couplings meet — if at all? That then determines $\alpha_5 = \bar{\alpha}_1(M_{CUT}) = \bar{\alpha}_2(M_{CUT}) = \bar{\alpha}_3(M_{CUT})$

Note I used $\hat{\alpha}_1 = \frac{5}{3}\alpha_1$.

Exercise: try this? (see my attempt, next page)

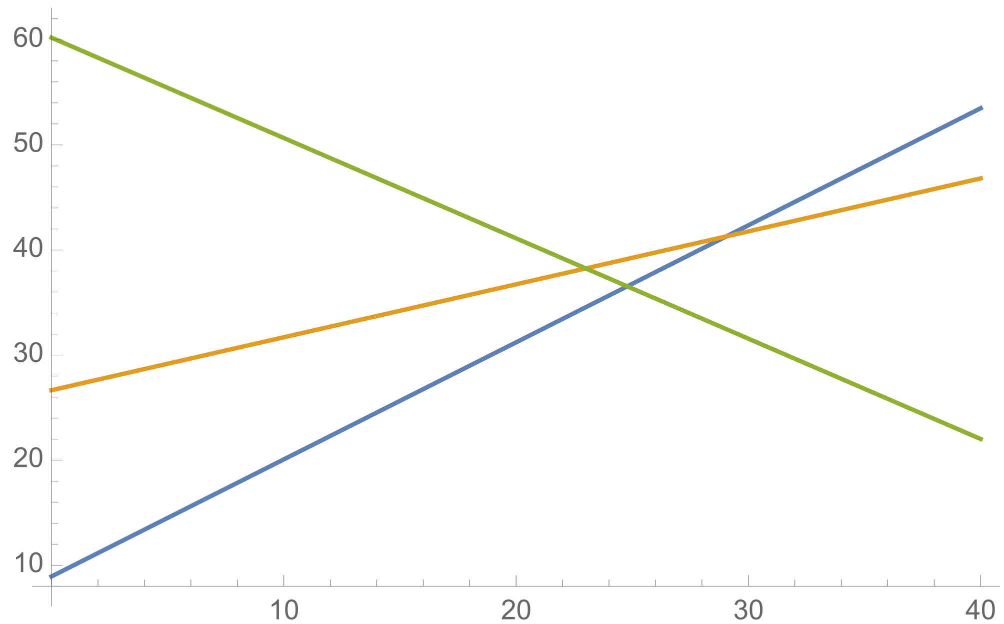
Use $\bar{\alpha}_3(M_Z) = 0.11$, and for α_1 & α_2 recall $e = \frac{g_1g_2}{\sqrt{g_1^2+g_2^2}} = g_1\cos\theta$

and $\cos\theta = \frac{M_W}{M_Z} \approx \frac{80}{90}$ and $\frac{e^2}{4\pi}(M_Z) = \alpha_{em}(M_Z) \approx \frac{1}{127}$

Plot linear functions: $\frac{1}{\bar{\alpha}_i(t)} = \frac{1}{\bar{\alpha}_i(M_Z)} + \frac{b_0^i}{2\pi}t \quad (t = \ln\mu/M_Z)$

In[14]:=
```
Plot[{a[7, 0.112], a[19 / 6, 1 / 127 / (1 - (8 / 9) ^ 2)], a[-6, 5 / 3 * 1 / 127 / ((8 / 9) ^ 2)]},
  {t, 0, 40}]
```

Out[14]=



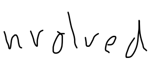
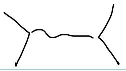
Exercise: Consider the nHDM = n Higgs doublets extension of the SM (ie, SM is n=1). Do the coupling constants tend to unify better than in the SM for some values of n?

Note that

$$\bar{\alpha}_i(t) = \alpha_5 \frac{1}{1+\frac{c_i}{2\pi}\alpha_5 t} = \alpha_5 \sum_{n=0}^{\infty} \left(\frac{b_0^i}{2\pi} \alpha_5 \ln \frac{M_{GUT}}{\mu} \right)^n$$

$\Rightarrow$ the RGE has summed up the $(\alpha \ln M)^n$ terms. This is called a "Leading-Log" (LL) resummation.

Note also that

(i) "matching" involved  $\Big|_{GUT}$ =  $\Big|_{SM}$ $\to$ tree level

(ii) "running" involved 1-loop beta function (from )

This is fairl common, matc = tree, run = loop.

How do we improve approximation?

(i) Match 1-loop $\Rightarrow$ $g_i = g_5(1 + \# g_5^2)$ at $\mu = M_{GUT}$

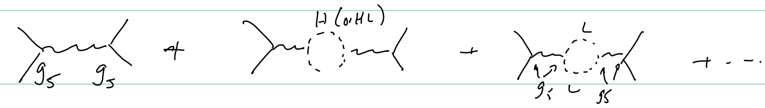
(ii) Run at 2-loops $\Rightarrow$ $\beta_i = \frac{b_0^i}{16\pi^2} g_i^3 + \frac{b_1^i}{(16\pi^2)^2} g_i^5$

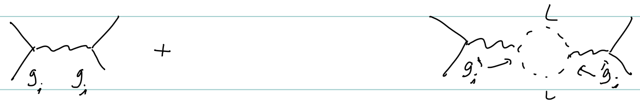
This formally gives $\bar{\alpha}_i(t) = \alpha_5 \left[\sum_n \left(\frac{b_0^i}{2\pi} \alpha_5 \ln \frac{M_{GUT}}{\mu} \right)^n + \alpha_5 \sum_n d_n \left(\alpha_5 \ln \frac{M_{GUT}}{\mu} \right)^n \right]$

ie, includes next-to-leading log (NLL) resummation

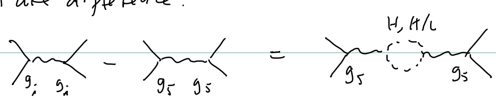
Ex: show this! (solve RGE to 2-loops)

For the 1-loop matching, diagramatically

$\mathcal{M}_S =$ 

$\mathcal{M}_{SM} =$ 

Take difference.



the light loop cancells out since at this order $g_i = g_S$

Once the leading term in energy$/M_H$ is selected (ie, subleading terms dropped) we obtain:

$$g_i^2 - g_S^2 = g_S^4 \left(\ln(\sim 1) + \text{constant} \right).$$